THE TWENTIETH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twentieth summer meeting and seventh colloquium of the Society were held at the University of Wisconsin during the week September 8–13, 1913. The attendance, which exceeded that of any previous summer meeting, included the following fifty-seven members:

Professor R. C. Archibald, Professor R. P. Baker, Professor G. N. Bauer, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Professor G. A. Bliss, Professor Oskar Bolza, Dr. H. T. Burgess, Professor W. H. Bussey, Professor R. D. Carmichael, Dr. G. R. Clements, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Dr. F. T. H'Doubler, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Dr. Dunham Jackson, Dr. A. J. Kempner, Professor Kurt Laves, Professor G. H. Ling, Professor A. C. Lunn, Professor H. W. March, Professor Max Mason, Mr. J. S. Mikesh, Professor C. N. Moore, Professor E. H. Moore, Professor W. F. Osgood, Professor B. L. Remick, Professor R. G. D. Richardson, Professor W. J. Risley, Professor W. H. Roever, Dr. J. E. Rowe, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Mr. T. M. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor E. R. Smith, Dr. E. B. Stouffer, Professor E. J. Townsend, Professor A. L. Underhill, Dr. S. E. Urner, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor R. E. Wilson, Professor R. M. Winger, Professor H. C. Wolff, Professor B. F. Yanney, Professor Alexander Ziwet.

The four sessions of the summer meeting proper occupied the first two days of the week. The first session opened with an address of welcome by Professor C. S. Slichter on behalf of the University of Wisconsin and the local members of the Society. At this session, and also at the final session, the President of the Society, Professor E. B. Van Vleck, occupied the chair. Professor Bolza presided at the second, and Professor Osgood at the third session. The Council an-

nounced the election of the following persons to membership in the Society: Mr. W. E. Anderson, University of Minnesota; Professor W. O. Beal, Illinois College; Dr. C. A. Fischer, Columbia University; Professor A. E. Landry, Catholic University of America; Lieutenant Salih Mourad, Ottoman Navy; Miss E. A. Weeks, Mount Holyoke College. Thirteen applications for membership in the Society were received.

The Secretary reported that Columbia University had provided a separate office for the Society and that under the new plan of administration the work of the Secretary, Treasurer, Librarian, Committee of Publication, and the sales department would now be handled in this one office with the assistance of a clerk.

Professor L. E. Dickson was elected Editor-in-Chief of the *Transactions*. It was decided to hold the summer meeting of 1915 at San Francisco, in connection with the Panama Exposition.

The Register of the Society will hereafter be issued at intervals of two or three years. In the intervening years only a list of officers and members will be published.

The arrangements made by the local committee for the comfort and entertainment of the members throughout the week were most hospitable. No place in the middle west could be more ideal for such a series of meetings than Madison. The spacious lecture halls, the beautiful campus occupying an elevated position overlooking the capitol building and the adjacent lakes, Mendota and Monona, the commodious University Club, used as headquarters, and the hospitality of President Van Vleck and other members of the faculty who opened their homes for the entertainment of the members, —these and many other items contributed to the success of the farthest west summer meeting, except that at St. Louis (1904), and the only western colloquium of the Society.

On Monday evening, Professor Van Vleck entertained at dinner the members of the Council and the colloquium lecturers. On Wednesday afternoon the committee provided a two-hours' special excursion on Lake Mendota, ending at the Golf Club House in time for the dinner, at which fifty-five persons gathered. President Van Vleck, acting as toastmaster, called upon Professors Osgood, Bolza, Moore, Blichfeldt, Dickson, and Dr. Jackson for informal talks, and Professor Slaught read a telegram to be sent to the Secretary,

expressing appreciation of his services to the Society and great regret at his enforced absence. At the close of the dinner Professor Ziwet voiced the unanimous sentiment in expressing thanks to the University and to the committee on arrangements for their generous hospitality. The dinner was followed by a moonlight ride on the lake back to the University Club. On Thursday the members were conducted by Professor Skinner about the campus and buildings of the University; and on Friday an automobile ride, provided by the mathematical faculty of the University and their friends, gave the members a fine opportunity to see the immediate surroundings of Madison. This ended in a most enjoyable buffet dinner at the home of President Van Vleck.

An account of the colloquium will appear later in the BULLETIN.

The following papers were presented at the four sessions of the summer meeting.

(1) Dr. E. B. LYTLE: "Note on iterable fields of integration."

(2) Professor W. H. BUSSEY: "The tactical problem of Steiner."

(3) Dr. JOSEPHINE E. BURNS: "The abstract definitions of the groups of degree eight."

(4) Professor WILLIAM MARSHALL: "The functions of the parabolic cylinder."

(5) Professor L. C. KARPINSKI: "The algorism of John Killingworth."

(6) Professor R. D. CARMICHAEL: "On series of iterated linear fractional functions."

(7) Professor R. D. CARMICHAEL: "Some theorems on the convergence of series."

(8) Mr. T. E. MASON: "The character of the solutions of certain functional equations."

(9) President E. B. VAN VLECK and Dr. F. T. H'DOUBLER: "On certain functional equations."

(10) Professor OSKAR BOLZA: "On the so-called 'abnormal' case of Lagrange's problem in the calculus of variations."

(11) Professors E. R. HEDRICK and W. D. A. WESTFALL: "An existence theorem for implicit functions."

(12) Professor R. G. D. RICHARDSON: "A solution of the Rayleigh minimum problem in the theory of sound."

(13) Professor G. C. EVANS: "The Cauchy problem for integro-differential equations."

(14) Dr. DUNHAM JACKSON: "A formula of trigonometric interpolation."

(15) Mr. J. W. ALEXANDER II: "Proof of the invariance of certain constants of analysis situs."

(16) Dr. J. E. ROWE: "On Fermat's theorem and related theorems (first paper)."

(17) Dr. J. E. ROWE: "On Fermat's theorem and related theorems (second paper)."

(18) Professor MAXIME BÔCHER: "The infinite regions of various geometries."

(19) Professor W. F. OSGOOD: "On functions of several variables which are meromorphic or analytic at infinity."

(20) Professor W. F. OSGOOD: "Note on line integrals on an algebraic surface $f(x, y, z) = 0$."

(21) Professor E. H. MOORE: "On a class of continuous functional operations associated with the class of continuous functions on a finite linear interval" (preliminary communication).

(22) Mr. A. R. SCHWEITZER: "On a general category of definitions of betweenness."

(23) Mr. A. R. SCHWEITZER: "The theory of linear vectors in Grassmann's extensive algebra."

(24) Mr. A. R. SCHWEITZER: "Remarks on functional equations."

(25) Mr. A. R. SCHWEITZER: "The general logical significance of uniformity of convergence of series."

(26) Professor EDWARD KASNER: "On the ratio of the arc to the chord for analytic curves."

(27) Dr. E. L. DODD: "The arithmetic mean as approximately the most probable value a posteriori under the Gaussian law."

(28) Professor E. J. WILCZYNSKI: "On the surfaces whose directrix curves are indeterminate."

(29) Professor J. B. SHAW: "On the transverse of a linear vector operator of n dimensions."

(30) Professor FLORIAN CAJORI: "Zeno's arguments on motion."

(31) Professor O. E. GLENN: "Note on a translation principle connecting the invariant theory of line congruences with that of plane n-lines."

(32) Professor F. R. SHARPE: "Conics through inflections of self-projective quartics."

(33) Professor F. R. SHARPE and Dr. C. F. CRAIG: "Plane curves with consecutive double points."

(34) Dr. MILDRED L. SANDERSON: "A method of constructing binary modular covariants."

(35) Dr. H. M. SHEFFER: "Superpostulates: introduction to the science of deductive systems."

(36) Dr. H. M. SHEFFER: "A set of six independent postulates for Boolean algebras."

(37) Professor R. M. WINGER: "Self-projective rational sextics."

(38) Professor R. M. WINGER: "Self-projective rational septimics" (preliminary report).

(39) Professor M. FRÉCHET: "Sur la notion de différentielle d'une fonction de ligne."

(40) Professor KURT LAVES: "A new theorem concerning the motion of two satellites of finite masses circulating in nearly commensurable motions of type $\frac{1}{2}$ about a central and homogeneous body of ellipsoidal shape."

(41) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups (fifth paper)."

(42) Professor T. R. RUNNING: "Graphical solutions of differential equations between two variables."

(43) Professor R. P. BAKER: "The genus of a group."

(44) Professor R. P. BAKER: "The topological configurations occurring in finite geometries."

(45) Professor R. D. CARMICHAEL: "On Fermat's theorem and certain related theorems."

(46) Professor H.W. MARCH: "Integral and series representations of an arbitrary function in terms of spherical harmonics."

Dr. Burns's paper was communicated to the Society through Professor Miller; Professor Fréchet's through Professor Dickson. Mr. Mason was introduced by Professor Carmichael; Dr. Sanderson by Professor Dickson. The paper of Professor Bôcher was read by Professor Osgood. The papers of Dr. Lytle, Miss Burns, Mr. Alexander, Dr. Dodd, Professors Marshall, Karpinski, Evans, Cajori, Glenn, Sharpe, Kasner, Running and Baker, the third paper of Professor Carmichael, and all the papers of Mr. Schweitzer except the first were read by title. Abstracts of the papers follow, with numbers corresponding to the titles in the above list.

1. In a former paper in the *Transactions*, volume 11 (1910), page 25, Dr. Lytle showed that a property, which he defined

as iterability of the field of integration, was a sufficient condition for a certain integral inequality. In the present note he shows that iterability is also necessary, and thus finds a necessary and sufficient condition that all limited functions simultaneously satisfy this inequality.

2. In this BULLETIN, volume 16, pages 19–22, Professor Bussey proved the existence of an arrangement in triads, tetrads, etc., for every number n of the form $2^j - 1$. In the present paper he proves the same thing in a more simple way and gives a method by which the triads, tetrads, etc., can easily be written down in any particular case.

3. Dr. Burns points out that the symbolic definitions of the abstract groups which may be represented as substitution groups on a small number of letters are effective in displaying the fundamental abstract properties of that important class of groups. The problem of finding the abstract definitions of the substitution groups of degree eight necessarily involves the detailed study of a large number of distinct groups, but the work may be materially simplified by considering several larger categories, each of which embraces several of the desired definitions. Such categories are represented by all groups which may be generated by two operators satisfying the following fundamental conditions:

1. $s_1^4 = s_2^4, \quad (s_1 s_2)^2 = 1, \quad s_1^{-1} s_2 s_1 = s_2^{\beta} s_1^{\alpha};$
2. $s_1^4 = s_2^4, \quad (s_1 s_2)^2 = 1, \quad s_1^2 s_2^2 = s_2^2 s_1^2;$
3. $s_1^4 = s_2^4, \quad (s_1 s_2)^3 = 1, \quad (s_1^2 s_2)^2 = 1;$
4. $s_1^6 = s_2^6, \quad (s_1 s_2)^2 = 1, \quad s_1^3$ and s_2^3 invariant;
5. $s_1^6 = s_2^6, \quad (s_1 s_2)^2 = 1, \quad s_1^2 s_2 = s_2 s_1^2;$
6. $s_1^8 = s_2^4, \quad (s_1 s_2)^2 = 1, \quad s_1^2 s_2 = s_2 s_1^2.$

In each case the equations given define a small category of abstract groups of finite order, many of which may be represented as substitution groups on eight or a smaller number of letters.

4. It is the object of Professor Marshall's paper to place the so-called functions of the parabolic cylinder on the same footing, as far as the possibility of numerical calculation is

concerned, as Bessel's functions, for example. To this end power series valid in the entire plane are deduced, which series contain only a single parameter, namely the ratio of the two constants κ and λ which appear in the defining equation

$$\frac{d^2U}{du^2} + (\kappa^2 u^2 + \lambda)U = 0.$$

Then semi-convergent expansions suitable for computation when the argument is large are developed, and the connection between these and the convergent series established. Finally, formulas for the roots of these functions are deduced. The paper is to appear in the *Archiv der Mathematik und Physik*.

5. Prominent among the English mathematicians and astronomers of the fifteenth century was John Killingworth of Merton College, Oxford. Professor Karpinski presents a study of Killingworth's algorism, which is preserved in manuscript Ee. III. 61, Cambridge University Library. This work, which explains our present system of arithmetic, was written in 1444 A. D. From the standpoint of the history of method and the development of science in England this is an important document. The study shows that the technique of the arithmetical operations as taught in Oxford had improved but little in the centuries between Adelard of Bath (c. 1125) and Killingworth. A comparison with the writings of contemporaries, such as Peurbach of Vienna and de Beldamandi of Italy, indicates that much greater progress in the art of arithmetic had been made on the continent. Doubtless the faulty instruction in arithmetic contributed to the general decline of mathematical science in England, which continued until the time of Newton.

This paper will appear in the *English Historical Review*.

6. Among the most important series known to mathematical analysis are the two classes of power series (ascending and descending) and the factorial series. Closely associated with the latter are the so-called binomial coefficient series. In his first paper Professor Carmichael introduces a common generalization of descending power series and factorial series and also of ascending power series and binomial coefficient series. These are the series of iterated linear fractional functions

$$\alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x) \cdots S_n(x)},$$

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n S_1(x) S_2(x) \cdots S_n(x),$$

where

$$S_1(x) = \frac{ax+b}{cx+d},\ ad-bc \neq 0; \quad S_k(x) = \frac{aS_{k-1}(x)+b}{cS_{k-1}(x)+d},\ k > 1,$$

a, b, c, d being constants. For the case where the substitution $x' = S_1(x)$ has two double points and the multiplier of the substitution is in absolute value equal to unity, the theory of both series is altogether exceptional; and consequently this case is not treated in detail. The general case subdivides into several special cases in each of which the general theory is essentially different from that in all the others. In each of these cases the fundamental elements of a general theory are developed. The nature of the region of convergence, of absolute convergence, of conditional convergence, and of uniform convergence is determined in every case; precise formulas also are obtained for the magnitude of these regions in terms of the coefficients of the series. It should be noted that the general results are in some respects in marked contrast to the simpler ones for the special cases which have been investigated heretofore.

7. No general criteria are known for the study of the convergence of a series of complex terms when the series does not converge absolutely. The object of Professor Carmichael's second paper is to generalize one of the few special methods which are already available for such study. Typical of the theorems obtained is the following: Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits

$$\lim_{n=\infty} S_n^{(k)}, \quad \lim_{n=\infty} nS_n^{(k-1)}, \quad \lim_{n=\infty} n^2 S_n^{(k-2)}, \quad \cdots, \quad \lim_{n=\infty} n^{k-1} S_n^{(1)}$$

exists and is finite, where

$$S_n^{(1)} = \sum_{i=1}^{n} a_i; \quad S_n^{(t)} = \sum_{i=1}^{n} S_i^{(t-1)}, \quad (t > 1).$$

Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of numbers such that $|\Delta^k c_1| + |\Delta^k c_2| + |\Delta^k c_3| + \cdots$ is convergent, where Δ is the difference symbol of the difference calculus. Then the series $a_1c_1 + a_2c_2 + a_3c_3 + \cdots$ is convergent (but not necessarily absolutely convergent).

8. Mr. Mason studies the solutions of the equations

$$\varphi(x+1)=\frac{A(x)\varphi(x)+B(x)}{C(x)\varphi(x)+D(x)} \quad \text{and} \quad \varphi(qx)=\frac{A(x)\varphi(x)+B(x)}{C(x)\varphi(x)+D(x)},$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ are rational, with respect to their rational, algebraic non-rational, algebraically transcendental, and transcendentally transcendental character. A partial treatment of these problems has been given by Tietze and Stridsberg. In the present paper methods are developed for finding all possible rational and algebraic non-rational solutions. For the case where $C(x) \not\equiv 0$ and there are no algebraic solutions, if we exclude a certain well-defined and quite restricted class of equations of exceptional character, then a necessary and sufficient condition that an equation of either of the types above have an algebraically transcendental solution is that a certain third order difference equation have a rational solution. The author shows how to find all the possible rational solutions of this equation. These algebraically transcendental functions also satisfy a certain Riccati differential equation with rational coefficients. These functions are asymptotic to the formal solutions of the equation in power series in $1/x$ or in $1/\sqrt{x}$. Equations are given which have no algebraic solutions but which do have algebraically transcendental solutions. The author finds a necessary and sufficient condition that an equation of the forms considered has only transcendentally transcendental solutions. A list of comprehensive classes of equations which have only transcendentally transcendental solutions is given. In the case when $A(x) \equiv D(x) \equiv 0$, there is a simple necessary and sufficient condition for algebraic solutions; if there are no algebraic solutions, then all the solutions are transcendentally transcendental.

9. The paper of Professor Van Vleck and Dr. H'Doubler is devoted to a treatment of the functional equations

$$\phi(x+y)\phi(x-y) = \phi^2(x)\phi^2(y) - \psi^2(x)\psi^2(y),$$
$$\psi(x+y)\psi(x-y) = \psi^2(x)\psi^2(y) - \phi^2(x)\phi^2(y).$$

From these equations the properties of the Jacobi ϑ-functions can be rapidly unrolled. The only real solutions not trivial

are, except for certain constant factors, pairs of ϑ-functions, or degenerate cases of the latter, such as the sine and cosine. In the complex plane there exist solutions of the functional equations which are non-analytic.

10. In his classical paper on Lagrange's multiplier rule, A. Mayer met with a certain exceptional case, which has since received the name of the "abnormal case of Lagrange's problem" (anomaler Fall, Hahn). In the present paper, Professor Bolza gives a detailed discussion of this abnormal case for the following generalized Lagrange-Mayer problem: To minimize the expression

$$U = \int_{t_0}^{t_1} f(y_1, \cdots, y_n, y_1', \cdots, y_n')dt + G(y_{10}, \cdots, y_{n0}; y_{11}, \cdots, y_{n1})$$

with respect to the totality of curves

$$y_i = y_i(t) \quad (t_0 \leqq t \leqq t_1; \quad i = 1, 2, \cdots, n),$$

which satisfy p differential equations

$$\varphi_\alpha(y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (\alpha = 1, 2, \cdots, p),$$

q finite equations

$$\psi_\beta(y_1, \cdots, y_n) = 0 \quad (\beta = 1, 2, \cdots, q),$$

and r initial conditions

$$\chi_\gamma(y_{10}, \cdots, y_{n0}; y_{11}, \cdots, y_{n1}) = 0 \quad (\gamma = 1, 2, \cdots, r),$$

$(y_{10}, \cdots, y_{n0})$ and $(y_{11}, \cdots, y_{n1})$ denoting the coordinates of the two end points of an admissible curve.

11. In this paper Professors Hedrick and Westfall prove a series of theorems on the existence of the solutions of implicit equations without any assumption concerning the existence of any derivative. The theorems cover the existence of solutions of single equations for one unknown, and of systems of equations; and they yield the classical theorems in those cases in which the derivatives exist. The fundamental assumption in the case of one equation $f(x, y) = c$ is that the difference quotient $[f(x, y) - f(x_0, y_0)]/[y - y_0]$ be definite in a region about the fixed point (x_0, y_0).

12. What is the form of that plane region of unit area which furnishes a minimum value to the particular characteristic parameter value λ of the differential equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\lambda z=0$$

which corresponds to a solution vanishing on the boundary? In other words, what is the shape of that drum head of prescribed area which emits the lowest fundamental tone? By means of a method similar to that used in simpler problems of the calculus of variations, Professor Richardson shows to be correct the surmise of Lord Rayleigh that the region must be circular. The desired solution of the differential equation is the solution of the following minimum problem: Of all functions $z(x, y)$ which vanish on the boundary C of a unit region R in the plane and which satisfy the condition

$$\int_R z^2 dR = 1,$$

which function z and which region R give to the integral

$$\int_R \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right\} dR$$

a minimum value? The first necessary condition furnishes an equation which shows that the greater the curvature of C, the greater the normal derivative of z. From this fundamental formula the theorem is derived.

13. Professor Evans extends the Cauchy existence theorem for partial differential equations to integro-differential equations of so-called "static type," i. e., equations in which the variables of differentiation are different from those of integration. The method of generalization adopted, which is that of the algebra of permutable and non-permutable functions, permits the partial extension of the idea of characteristics to equations of this kind. These curves are given in terms of the variables of differentiation, and are called ordinary characteristics. For special equations every curve in the plane may be an ordinary characteristic; but equivalent equations may sometimes be found whose characteristics are defined. If a curve is not tangent to an ordinary character-

istic, analytic values of the function and its proper derivatives, assigned on the curve, will determine an analytic solution; along a characteristic these values may not be assigned arbitrarily, and do not by themselves determine a unique solution.

14. The ordinary formula of trigonometric interpolation for a function $f(x)$ of period 2π gives a trigonometric sum of order n which coincides in value with $f(x)$ at $2n + 1$ points evenly distributed over a period. The formula is closely analogous to that for the partial sum of the Fourier series for $f(x)$, and the convergence properties of the interpolating function, as n becomes infinite, are similar to those of the Fourier sum. In particular, the mere continuity of $f(x)$ is not a sufficient condition for convergence. Dr. Jackson shows that an interpolating function suggested by Fejér's arithmetic mean in the theory of Fourier's series does always converge, uniformly, if $f(x)$ is continuous; it is a trigonometric sum of order $n - 1$, taking the values of the given function at n equidistant points. The formula which he introduced, for the sake of the rapidity of its convergence in certain cases, in a paper presented at the February meeting of the Society, also converges for every continuous function $f(x)$, but does not coincide exactly with $f(x)$ at the given points, as the present one does. The new formula is examined as to its convergence for certain classes of discontinuous functions; for convergence at any one point, for example, it is sufficient that $f(x)$ be continuous at that point and remain finite everywhere. A simple transformation gives a formula of polynomial interpolation with given points unequally spaced.

15. It has been shown by Poincaré that if two complexes which define the same manifold be made up of a finite number of analytic pieces, their Betti numbers and coefficients of torsion are the same. In this paper, Mr. Alexander proves the invariance of the Betti numbers, the coefficients of torsion, and the numbers which are obtained by reducing the matrices of the complex modulo 2, without assuming the analytic character of the complexes.

16. It is the purpose of Dr. Rowe's first paper to present and illustrate a method of attacking certain problems in regard

to the sums of powers of integers. The four important theorems follow.

(1) The difference of the nth powers (n an odd number > 1) of two odd integers cannot be the kth power (k an integer $\geqq n$) of an integer if the difference of the odd integers is different from zero mod 2^n.

(2) The sum of the nth powers of two odd integers cannot be the kth power of an integer if the sum of the odd integers is different from zero mod 2^n.

(3) The difference of the nth powers of two odd numbers, one of the form $4r + 1$, and the other of the form $4r - 1$, cannot be the k'th power (k' an integer > 1) of an integer.

(4) The sum of the nth powers of two odd numbers, both of the form $4r + 1$, or both of the form $4r - 1$, cannot be the k'th power of an integer.

Applied to Fermat's theorem, theorems (3) and (4) above invalidate one half of all possible solutions of the equation $x^n + y^n = z^n$. Theorems (1) and (2) together with the work of Kummer are shown to prove the impossibility of a solution of $x^n + y^n = z^n$ in integers for all values of n so long as $z(z > x, z > y)$ is a number not greater than 10^{29}.

Further, the theorems of the paper are important quite apart from their application to Fermat's theorem.

This paper has been published in the Johns Hopkins Circular, July, 1913.

17. Dr. Rowe's second paper is concerned with the applications of the theorems of the first paper to more restricted sets of integers. Possibly the most important result is that the first two theorems of the first paper together with Dickson's work (*Quarterly Journal*, volume 40) show that there is no solution of the equation $x^n + y^n = z^n$ in integers x, y and z all prime to n for any value of n if z (the greatest of the integers) is an integer of not more than 2,063 digits.

The one theorem of the paper which is in a different class from the others is: The sum of even powers of two odd integers cannot be the Sth power (S an integer > 1) of an integer.

18. In plane and solid projective geometry the infinite region is a line and a plane respectively; in the real geometry of inversion, a point. Much less familiar are the infinite re-

gions for other geometries, for instance in the geometry of inversion in the complex plane. Here the infinite region consists of two ideal minimal lines, one of each set; and in the geometry of inversion of complex three dimensional space it consists of a minimal cone. In the real or complex plane of analysis (cf. Osgood, *Transactions*, volume 13 (1912), page 159) the infinite region consists of the two real straight lines $x = \infty$ and $y = \infty$; and in the three dimensional space of analysis, of three planes. These facts are elaborated in Professor Bôcher's paper, and the attempt is made to make it clear that these infinite regions are just as useful in their proper places as the more familiar line and plane at infinity of projective geometry in theirs.

19. Professor Osgood's first paper contains the following theorems.

Theorem A. If $f(z_1, \cdots, z_n)$ is analytic in every point of the coordinate axes in the space of analysis, then f is a constant.

Theorem A'. If $f(z_1, \cdots, z_n)$ is analytic in those points of the infinite region of the space of analysis which correspond to any $n - 1$ north poles combined with any point whatever of the nth sphere, then f is a constant.

Corollary. If $f(z_1, \cdots, z_n)$ is analytic in every point of the infinite region of the space of analysis, then f is a constant.

Theorem B. If $f(z_1, \cdots, z_n)$ is analytic at all points outside a fixed hypersphere

$$x_1^2 + y_1^2 + x_2^2 + \cdots + y_n^2 > G,$$

and if f is finite in this region, then f is a constant.

20. Professor Osgood's second paper calls attention to the behavior at infinity of integrals $\int (P dx + Q dy)$ on an algebraic surface $f(x, y, z) = 0$, which are everywhere finite there. Some integrals of this class are analytic at infinity in the space of analysis, others are analytic at infinity in projective space.

21. The purpose of Professor E. H. Moore's preliminary communication is to indicate an instance of his general Fredholm theory of linear integral equations on the basis Σ_4 (cf. BULLETIN, April, 1912) in which the independent variable p

of the functions involved is itself a function π, the functions thus becoming functional operations on the variable function π.

22. In the *American Journal*, volume 31 (1909), page 396, is given a definition of "α is between $\beta\gamma$" in terms of sameness of sense of tetrahedra. On page 388, l. c., is given an analogous definition based on the relation righthandedness of tetrahedra. On page 366, l. c., it is noted that in addition to the relations "sameness of sense" and "righthandedness" the relations $\alpha_1\alpha_2 S\beta_1\beta_2 \cdots \beta_p$, $\alpha_1\alpha_2 \cdots \alpha_p B\beta_1\beta_2$, and $\alpha_1 I\beta_1\beta_2 \cdots \beta_{p+1}$ also generate n-dimensional geometry. This is evident at once, if it is recognized that corresponding definitions of betweenness are given, by employing as a guide the definition in terms of sameness of sense. Mr. Schweitzer points out that the five definitions of betweenness indicated have the common characteristic that they define affirmatively a linear conception in terms of a spatial conception. The problem of phrasing corresponding geometric systems so that the basal relations function most naturally in each instance is considered in a memoir "A theory of geometrical relations" which is not yet concluded.

23. Using his "Foundations of Grassmann's extensive algebra," *American Journal* (1913), pages 37–56, as a basis Mr. Schweitzer deduces in detail the theory of linear vectors, The first part of the paper is concerned with the deduction of the relation $\beta - \beta' = q(\alpha - \alpha')$, where q is any number, and its fundamental properties; a feature is the use of the author's definition of parallel lines, l. c. In the second part the relation

$$(a + b + c + d)\xi = a\alpha + b\beta + c\gamma + d\delta,$$

where a, b, c, d are any numbers, is derived. In the third part the regressive product of two vectors is defined, also their metrical properties such as congruence, length, etc. The concluding part investigates to what extent the author's fundamental properties (l. c.) induce the equality of domains of points. This part represents essentially an application of Grassmann's statement:

"Die Gleichheit zweier Theile eines Elementarsystems besteht im allgemeinsten Sinne darin, dass beide von dem

in einfachem Sinne erzeugten Systeme von Elementen gleiche Gebiete umfassen, nämlich so, dass wechselseitig jedem Element des einen Gebietes ein, aber auch nur ein Element des andern entspricht.

24. In Mr. Schweitzer's third paper the following theorems on Grassmann's synthetic and analytic associations are proved:

I. $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $(y \cap z) = (z \cap y)$ and $\{y \cup (y \cup z)\} = z$.

II. $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$ and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$.

III. $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$ and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$.

As an application, theorems on the author's functional relations are obtained by replacing the associations $(x \cap y)$ and $(x \cup y)$ by $\phi(x, y)$ and $f(x, y)$ respectively. Finally, the problem of finding the quasi-transitive function $f_1(x, y)$ such that

$$f_1\{f(x, y), z\} = f\{f_1(x, z), f_1(y, z)\},$$

where $f(x, y)$ is quasi-transitive, is considered. An actual and not merely formal solution is obtained by using certain well known functional relations due to Abel and Schroeder.

25. The uniformity of convergence of series is justly considered one of the most important conceptions that underlie analysis. Mr. Schweitzer first reviews the manner in which this conception enters into various analytic disciplines; then discusses its relation to the general concept *iteration* and finally takes account of uniformity of convergence by means of his "principle of continuation," which is stated thus:

"The existence of a class of particular elements (or operands) affected by particular operations implies the existence of a class of general elements affected by general operations."

Two special applications of this principle must here be noted, viz., (1) invariance of domain of elements under generalization of operations; and (2) persistence of operational properties under generalization of operands.

26. Professor Kasner shows that for imaginary analytic curves the limit of the ratio of the arc to the chord may not be

unity. The limit is always real, and if it is not unity is at most equal to $\frac{2}{3}\sqrt{2} = .94\cdots$. A different result is obtained for irregular (singular) points. An extension is made to space.

27. Under the postulate that each real number is equally likely a priori to be the true value, the attempt has been made to deduce from the Gaussian law the principle that the arithmetic mean of n measurements is a posteriori the most probable value of the unknown true value. But this postulate has already* been shown to be untenable. Dr. Dodd investigates several postulates, and finds each of the following three postulates to be inadequate: (1) The a priori probability is constant in a certain interval, and is zero outside this interval. (2) The a priori probability is continuous, and is zero outside a certain interval. (3) The a priori probability is continuous for all real values and does not vanish.

But if the a priori probability is such that the derivative of its logarithm is numerically less than some constant for all real values, then a posteriori under the Gaussian law, after n measurements have been made, the arithmetic mean of these measurements differs from the most probable value by an infinitesimal, as n increases indefinitely. Other conditions less restrictive are found under which the arithmetic mean approximates the most probable value.

28. The linear complexes which osculate the two asymptotic curves of a surface point at their point of intersection have in common a linear congruence of whose directrices one passes through the surface point while the other lies in the corresponding tangent plane. If these directrices be constructed for every point of a surface, there arise two congruences, the directrix congruences of the surface. There exist, in general, two one-parameter families of curves on the surface such that the directrices of the points of such a curve form developables. Professor Wilczynski's paper is devoted to the exceptional case where these so-called directrix curves are indeterminate, i. e., where the directrices of all points of an arbitrary curve on the surface form developables. It is shown that every non-ruled surface of this kind is a projective transformation of an integral surface of the completely integrable system

* This BULLETIN, June, 1913, p. 481.

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\varphi_u}{\varphi}\frac{\partial\theta}{\partial u} + 2\varphi\frac{\partial\theta}{\partial v} = 0,$$

$$\frac{\partial^2\theta}{\partial v^2} + 2\varphi\frac{\partial\theta}{\partial u} + \frac{\varphi_v}{\varphi}\frac{\partial\theta}{\partial v} = 0,$$

where θ is a non-vanishing constant or any solution of the partial differential equation

$$\frac{\partial^2 \log\varphi}{\partial u\partial v} = 4\varphi^2 + \frac{k}{\varphi},$$

where k is a constant, which is or is not equal to zero according as the corresponding surface has asymptotic curves which do or do not belong to linear complexes. In the former case, the explicit equations of the surface can be found by quadratures.

One of the directrix congruences of any surface of this class consists of all of the lines through a fixed point, while the second directrix congruence is composed of all of the lines of a fixed plane. These are called the directrix point and directrix plane of the surface, and are united in position if and only if $k = 0$.

If the asymptotic curves of the surface are projected from the directrix point on the directrix plane, there is obtained a net of plane curves which has very remarkable properties. It is periodic of period 3 under Laplace transformation, and the two nets distinct from it obtained by Laplace transformation are so related as to have at each pair of corresponding points a common osculating conic. These properties are characteristic; i. e., any net of plane curves which has these properties may be regarded as a projection of the asymptotic curves of a surface with indeterminate directrix curves. A net of this kind is determined except for three arbitrary constants by means of one curve of each of its two families, and these may be assigned arbitrarily. There exists a single infinity of surfaces with indeterminate directrix curves which correspond to a given net of this kind and a given point as directrix point.

Incidentally a complete geometrical interpretation is obtained for the equality of the invariants h and k of a single partial differential equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta = 0,$$

a case which has received considerable attention especially from Darboux and Moutard, but whose geometrical significance apparently has not been discussed so far.

29. In this paper Professor Shaw calls attention to the necessity of considering the axes of a linear vector operator (which are foreign to a matrix). For the transverse the shear regions are made up of vectors which are conjugate to those defining the shear regions of the original linear vector operator, but respectively reversed in order. In the case of self-transverse operators when the multiplicity of any root is $m > 1$ the shear regions are representable in the forms

$$\psi(i_1 + \sqrt{-1}\,j_m), \quad \psi(i_2 + \sqrt{-1}\,j_{m-1}), \cdots$$
$$\psi(i_r + \sqrt{-1}\,j_r), \quad \psi(i_r - \sqrt{-1}\,j_r), \cdots$$
$$\psi(i_1 - \sqrt{-1}\,j_m).$$

In this $r = \frac{1}{2}m$ for m even, $\frac{1}{2}(m+1)$ for m odd. In case m is odd $j_r = 0$. The vectors i_s, j_s form a mutually orthogonal system, and ψ has an inverse.

30. Professor Cajori's paper is a study of the purpose of Zeno's arguments as it is reported by Plato and as it is set forth by modern writers, particularly by E. Zeller, W. Windelband, M. Cantor, V. Cousin, G. Grote, P. Tannery, G. Milhaud, P. Natorp, and J. Cohn.

31. Professor Glenn defines and applies a translation principle by which every invariant of a plane n-line corresponds to a contravariant of a definite line congruence in 3-space. The idea of a tangential equation of the congruence is developed from the invariantive point of view.

32. Ciani (*Rendiconti di Palermo*, 1899; *Annali di Matematica*, 1901) has discussed the systems of conics through 8 inflections on the various self-projective quartics. In this paper Professor Sharpe confirms Ciani's results by a different method and also proves the existence of systems of conics through 6 inflections,

the Klein quartic having 2100 such conics in addition to the 112 found by Ciani.

33. The existence of irreducible plane curves with any number $\leqq \frac{1}{2}(n-1)(n-2)$ of distinct double points was proved by Snyder (this BULLETIN, volume 15, page 1, 1908). Guccia (*Supplemento Rendiconti Palermo*, volume 5, page 1, 1910) proposed the problem of the maximum number of consecutive double points for an irreducible plane curve of order n. In this paper Professor Sharpe and Dr. Craig show that the maximum number is at least the greatest integer in $\frac{1}{4}(n^2+3n-14)$. The rational sextics with consecutive double points are shown to be projectively equivalent and their equation in a canonical form is determined.

34. In this paper Miss Sanderson shows a way of constructing modular covariants of a binary form similar to that in which ordinary covariants are constructed as symmetric functions of the roots. A form of the nth degree is written as the product of n linear factors, the ratios of whose coefficients are the roots of the form when it is equated to zero. It proves simpler to construct the covariants as functions of these coefficients rather than as functions of the roots. It is found that many different types of symmetric functions may enter in addition to the two simple types which suffice for the ordinary covariants. Some covariants are found to be easily obtainable, as in the ordinary theory, from the coefficient of the leading term.

35. By means of uniform sets of "superpostulates" Dr. Sheffer is enabled (1) to determine, by one type of formula, the class of deductive systems having a postulated property; (2) to determine completely, by the same type of formula, a given deductive system; (3) after the reduction of an ordinary postulate set to its corresponding superpostulate set, to solve by inspection (*a*) the problems of consistence, independence, and one-valuedness, (*b*) a new problem, that of internal independence.

36. Dr. Sheffer determines Boolean algebras by six independent postulates involving a non-commutative operation. The existence of the special element u is postulated; that of z and of a' is proved.

37. In a former paper, to be published in the *American Journal*, Professor Winger tabulated the varieties of self-projective rational curves of the fourth and fifth orders. In the present paper he considers the case of the rational sextic, exhibits the types associated with the different groups, and points out some of the more immediate inferences.

There are one or more projectively distinct curves attached to each of the following groups: cyclic G_n, dihedral G_{2n} ($2 \leqq n \leqq 6$), the tetrahedral G_{12}, the octahedral G_{24}, the icosahedral G_{60}, and an infinite group. Thus the sextic is the first curve to admit the tetrahedral group (simply) or the icosahedral, the group of maximum order for rational curves.

38. This is a continuation of the preceding paper. The following general theorem is proved: A rational plane curve C^m of odd order cannot admit the tetrahedral G_{12}, the octahedral G_{24}, the icosahedral G_{60}, nor a dihedral G_{2n}, n even. Combined with a former theorem, this limits the types of septimics to those invariant under cyclic groups of orders 2, . . . , 7 and dihedral groups of orders 6, 10, and 14. All types are found to occur.

39. Following J. Hadamard, Professor Fréchet criticizes the definition of derivatives in functional calculus given by V. Volterra. He shows that one definition of differentials must be given before that of derivatives, and gives a new definition of differentials in general analysis. He then proves that the differential of a function U_L of a line L can be written

$$\int_a^b (y_0'\Delta z - z_0'\Delta y)dp + \int_a^b (z_0'\Delta x - x_0'\Delta z)dq + \int_a^b (x_0'\Delta y - y_0'\Delta x)dr$$
$$+ R[x_0'(a)\cdot\Delta x(a) + y_0'(a)\cdot\Delta y(a) + z_0'(a)\cdot\Delta z(a)]$$
$$+ S[x_0'(b)\cdot\Delta x(b) + y_0'(b)\cdot\Delta y(b) + z_0'(b)\cdot\Delta z(b)],$$

where x_0', y_0', z_0' are the derivatives of the coordinates $x_0(t)$, $y_0(t)$, $z_0(t)$ on L_0, ($a \leqq t \leqq b$); $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$ are the projections of the displacement; A, B, C; A', B', C' are constants; $p(t)$, $q(t)$, $r(t)$ are functions with limited total fluctuation, and where the integrals are Stieltjes's integrals. The paper will appear in the *Transactions*.

40. Under the assigned conditions of Professor Laves's paper the system of two satellites is assumed to be placed at the time t_0 at such a distance from the central mass that the sum of the angles of recession of the nodes of the two orbits on the equatorial plane of the central mass equals twice the angle of elongation. The angles of inclination γ and γ' are considered small quantities of the first order at the time t_0. The following theorem is obtained: Even if the original inclinations of the orbits are small, by the mutual perturbations of the satellites additional terms are created of a librational character. The "imparted" inclination of each satellite contains four factors: (1) the mass of the perturbing satellite, (2) its inclination, (3) the ratio of the mean daily motion n of the perturbed body to the mean daily motion of the point of conjunction with the perturbing satellite, (4) $\frac{1}{1-\kappa}$, where $\kappa = \frac{\beta n + \beta' n'}{4n' - 2n}$; βn and $\beta' n'$ measure the recession of the line of nodes of each body upon the equator.

The close proximity of κ to unity makes the librational term in the case of one of Saturn's inner system of satellites of primary importance. The theorem explains the existence of considerable inclinations in this system, first discovered by H. Struve.

41. Let G be a primitive group of linear homogeneous substitutions of determinant unity in n variables. In Professor Blichfeldt's paper the following points are noted: (1) The order of an abelian subgroup of G of variety m is $\leqq 5^{m-1}$ if it is free from similarity substitutions. If the order is at the same time a power of a prime $p \neq 2$ or 5, then it is $\leqq 4^{n-1}$. (2) If the order of a Sylow subgroup of G be p^λ, where $p > 3$, $> n/2$, and $\neq n$, then that subgroup is abelian. (3) Let these subgroups be arranged as follows:

$\frac{1}{2}n < p_1 \leqq \frac{1}{2}n + 1$, the order being $p_1^{\lambda_1}$;

$\frac{1}{2}n + 1 < p_i \leqq n + 1$, the order being $p_i^{\lambda_i}$ $(i = 2, 3, \cdots, a;\ p_i \neq n)$;

$n + 1 < p_j$, the order being $p_j^{\lambda_j}$ $(j = a + 1, \cdots, b)$.

Then there is in G an abelian subgroup of order

$$p_1^{[\lambda_1]} \cdot \prod_{i=2}^{a} p_i^{\lambda_i - 1} \cdot \prod_{j=a+1}^{b} p_j^{\lambda_j},$$

where $[\lambda_1] = 0$ or $\lambda_1 - 1$, according as $\lambda_1 \leqq 2$ or > 2. (4) The order of G must be $< \frac{1}{2}n[(n+1)!] \cdot 5^{2(n-1)} \cdot 4^{g(n-1)}$, where g represents the number of primes that are $\geqq 2$ and $\leqq n/2$.

42. In Professor Running's paper different values are assigned to y in the equation

$$f(dy/dx, x, y) = 0$$

and curves corresponding to the resulting equations are traced with dy/dx as ordinate. Starting from the point represented by the initial condition, points on the curves are joined in such a way as to make the area under each connecting line equal to the difference between corresponding values of y. The derived curve is approximated to by the connecting lines. The coordinates of points on the integral curve are read from the figure. The method is applied to differential equations of the nth order where the initial conditions are represented by the first $n - 1$ derivatives at a given point.

The first part of this paper will appear in the November number of the *American Mathematical Monthly*.

43. In a previous paper Professor Baker showed that the genus of a Cayley diagram representing a group might fall below the Dyck genus if the group were abelian. In the present paper an example of reduction is shown to occur in the case of three generators not abelian, namely, the genus of G_{32}, Burnside, page 81, XIV, $m = 2$, can be reduced to 10, the Dyck genus being 13. For diagrams not having self-touching polygons the genus is not less than the sum of the genera of the generational "Nebengruppe," that is, those lines of a rectangular table of the group which are extensions of a subgroup by the power of the remaining generator.

44. Professor Baker's second paper is in abstract as follows: The $S(3, 2, 7)$ can be represented by 7 points, 21 arcs, and 14 triangles on an anchor ring. Seven of the triangles may have the points of the lines of the system, and the other seven

the points of the lines of another system; or the arrangement of the seven lines may appear as one triangle and six routes. This alternative is general for all the systems $S(s+1, 2, s^2+s+1)$. The $S(5, 2, 21)$ can be exhibited by a configuration of 21 pentagons, corresponding to the columns of the system, and 21 skew pentagons. The genus is 22 and the arrangement is invariant under a rotation of $2\pi/21$. It is possible so to choose the notation that the three real points and the two Galois imaginary points on each line are simultaneously separated.

The $S(s+1, 2, s^2+s+1)$, s an odd prime, is not representable by a simple M_2. The polygons representing the lines fall into at least two cycles at a point. For $s(4, 2, 13)$ the representation may be made by a body with one ordinary hole and thirteen holes only joined up at a pinch point. At these pinch points eight quadrilaterals come together in two sets of four, two of the system and two of an auxiliary system in each.

To represent further invariances under the group it is necessary to use more complicated manifolds. For example, tetrahedra hanging together on edges, in pairs in an R_3 not of simple connectivity in an R_4, are needed for the case $S(4, 2, 13)$.

45. In this paper Professor Carmichael proves several general theorems concerning the positive or negative integers x, y, z and the integers n and k (each greater than unity) which satisfy the equation

$$x^n - y^n = z^k.$$

Typical of these results is the following: the greatest divisor d of $x - y$ which is prime to x, y and n is a perfect kth power. In addition he proves the following special theorems: the difference of the fourth powers of two relatively prime integers cannot be the cube of an integer; neither the sum nor the difference of the sixth powers of two integers can be the square of an integer.

46. In the first part of Professor March's paper it is shown that the double integral representation of an arbitrary function in terms of spherical harmonics,

$$f(\theta) = \int_0^\infty \alpha d\alpha P_{\alpha-\frac{1}{2}}(\cos\theta) \int_0^\pi f(\gamma) S_{\alpha-\frac{1}{2}}(\cos\gamma) \sin\gamma d\gamma,$$

established in his Munich dissertation,* "Ueber die Ausbreitung der Wellen der drahtlosen Telegraphie auf der Erdkugel," for the case where $f(\theta)$ is continuous in the interval $(0, \pi)$ and is subject to certain other restrictions, is also capable of representing a function which has a finite number of finite discontinuities in this interval, and that at such a point of discontinuity the integral represents the mean of the two limiting values of the function.

In the second part of the paper use is made of the method employed in establishing the integral representation to obtain the well-known development of an arbitrary function in a series of spherical harmonics. The method has the advantage of not requiring a proof of the possibility of the development, as this is made to depend upon the possibility of the development of an arbitrary function in a Fourier series of special form.

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INTUITIONISM AND FORMALISM.†

BY DR. L. E. J. BROUWER.

(Inaugural address at the University of Amsterdam, read October 14, 1912.)

THE subject for which I am asking your attention deals with the foundations of mathematics. To understand the development of the opposing theories existing in this field one must first gain a clear understanding of the concept "science"; for it is as a part of science that mathematics originally took its place in human thought.

By science we mean the systematic cataloguing by means of laws of nature of causal sequences of phenomena, i. e., sequences of phenomena which for individual or social pur-

* See note in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 20 (1911), pp. 353–363.

† Translated for the BULLETIN by Professor ARNOLD DRESDEN.

poses it is convenient to consider as repeating themselves identically,—and more particularly of such causal sequences as are of importance in social relations.

That science lends such great power to man in his action upon nature is due to the fact that the steadily improving cataloguing of ever more causal sequences of phenomena gives greater and greater possibility of bringing about desired phenomena, difficult or impossible to evoke directly, by evoking other phenomena connected with the first by causal sequences. And that man always and everywhere creates order in nature is due to the fact that he not only isolates the causal sequences of phenomena (i. e., he strives to keep them free from disturbing secondary phenomena) but also supplements them with phenomena caused by his own activity, thus making them of wider applicability. Among the latter phenomena the results of counting and measuring take so important a place, that a large number of the natural laws introduced by science treat only of the mutual relations between the results of counting and measuring. It is well to notice in this connection that a natural law in the statement of which measurable magnitudes occur can only be understood to hold in nature with a certain degree of approximation; indeed natural laws as a rule are not proof against sufficient refinement of the measuring tools.

The exceptions to this rule have from ancient times been practical arithmetic and geometry on the one hand, and the dynamics of rigid bodies and celestial mechanics on the other hand. Both these groups have so far resisted all improvements in the tools of observation. But while this has usually been looked upon as something accidental and temporal for the latter group, and while one has always been prepared to see these sciences descend to the rank of approximate theories, until comparatively recent times there has been absolute confidence that no experiment could ever disturb the exactness of the laws of arithmetic and geometry; this confidence is expressed in the statement that mathematics is "the" exact science.

On what grounds the conviction of the unassailable exactness of mathematical laws is based has for centuries been an object of philosophical research, and two points of view may here be distinguished, *intuitionism* (largely French) and *formalism* (largely German). In many respects these two viewpoints

have become more and more definitely opposed to each other; but during recent years they have reached agreement as to this, that the exact validity of mathematical laws as laws of nature is out of the question. The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper.

In Kant we find an old form of intuitionism, now almost completely abandoned, in which time and space are taken to be forms of conception inherent in human reason. For Kant the axioms of arithmetic and geometry were synthetic a priori judgments, i. e., judgments independent of experience and not capable of analytical demonstration; and this explained their apodictic exactness in the world of experience as well as in abstracto. For Kant, therefore, the possibility of disproving arithmetical and geometrical laws experimentally was not only excluded by a firm belief, but it was entirely unthinkable.

Diametrically opposed to this is the view of formalism, which maintains that human reason does not have at its disposal exact images either of straight lines or of numbers larger than ten, for example, and that therefore these mathematical entities do not have existence in our conception of nature any more than in nature itself. It is true that from certain relations among mathematical entities, which we assume as axioms, we deduce other relations according to fixed laws, in the conviction that in this way we derive truths from truths by logical reasoning, but this non-mathematical conviction of truth or legitimacy has no exactness whatever and is nothing but a vague sensation of delight arising from the knowledge of the efficacy of the projection into nature of these relations and laws of reasoning. For the formalist therefore mathematical exactness consists merely in the method of developing the series of relations, and is independent of the significance one might want to give to the relations or the entities which they relate. And for the consistent formalist these meaningless series of relations to which mathematics are reduced have mathematical existence only when they have been represented in spoken or written language together with the mathematical-logical laws upon which their development depends, thus forming what is called symbolic logic.

Because the usual spoken or written languages do not in the least satisfy the requirements of consistency demanded of this symbolic logic, formalists try to avoid the use of ordinary language in mathematics. How far this may be carried is shown by the modern Italian school of formalists, whose leader, Peano, published one of his most important discoveries concerning the existence of integrals of real differential equations in the *Mathematische Annalen* in the language of symbolic logic; the result was that it could only be read by a few of the initiated and that it did not become generally available until one of these had translated the article into German.

The viewpoint of the formalist must lead to the conviction that if other symbolic formulas should be substituted for the ones that now represent the fundamental mathematical relations and the mathematical-logical laws, the absence of the sensation of delight, called "consciousness of legitimacy," which might be the result of such substitution would not in the least invalidate its mathematical exactness. To the philosopher or to the anthropologist, but not to the mathematician, belongs the task of investigating why certain systems of symbolic logic rather than others may be effectively projected upon nature. Not to the mathematician, but to the psychologist, belongs the task of explaining why we believe in certain systems of symbolic logic and not in others, in particular why we are averse to the so-called contradictory systems in which the negative as well as the positive of certain propositions are valid.*

As long as the intuitionists adhered to the theory of Kant it seemed that the development of mathematics in the nineteenth century put them in an ever weaker position with regard to the formalists. For in the first place this development showed repeatedly how complete theories could be carried over from one domain of mathematics to another: projective geometry, for example, remained unchanged under the interchange of the rôles of point and straight line, an important part of the arithmetic of real numbers remained valid for various complex number fields and nearly all the theorems of elementary geometry remained true for non-archimedian

* See Mannoury, "Methodologisches und Philosophisches zur Elementarmathematik," pp. 149–154.

geometry, in which there exists for every straight line segment another such segment, infinitesimal with respect to the first. These discoveries seemed to indicate indeed that of a mathematical theory only the logical form was of importance and that one need no more be concerned with the material than it is necessary to think of the significance of the digit groups with which one operates, for the correct solution of a problem in arithmetic.

But the most serious blow for the Kantian theory was the discovery of non-euclidean geometry, a consistent theory developed from a set of axioms differing from that of elementary geometry only in this respect that the parallel axiom was replaced by its negative. For this showed that the phenomena usually described in the language of elementary geometry may be described with equal exactness, though frequently less compactly in the language of non-euclidean geometry; hence, it is not only impossible to hold that the space of our experience has the properties of elementary geometry but it has no significance to ask for *the* geometry which would be true for the space of our experience. It is true that elementary geometry is better suited than any other to the description of the laws of kinematics of rigid bodies and hence of a large number of natural phenomena, but with some patience it would be possible to make objects for which the kinematics would be more easily interpretable in terms of non-euclidean than in terms of euclidean geometry.*

However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be

*See Poincaré, La Science et l'Hypothèse, p. 104.

repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i. e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units.

In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry, and not only for elementary two- and three-dimensional geometry, but for non-euclidean and n-dimensional geometries as well. For since Descartes we have learned to reduce all these geometries to arithmetic by means of the calculus of coordinates.

From the present point of view of intuitionism therefore all mathematical sets of units which are entitled to that name can be developed out of the basal intuition, and this can only be done by combining a finite number of times the two operations: "to create a finite ordinal number" and "to create the infinite ordinal number ω"; here it is to be understood that for the latter purpose any previously constructed set or any previously performed constructive operation may be taken as a unit. Consequently the intuitionist recognizes only the existence of denumerable sets, i. e., sets whose elements may be brought into one-to-one correspondence either with the elements of a finite ordinal number or with those of the infinite ordinal number ω. And in the construction of these sets neither the ordinary language nor any symbolic language can have any other rôle than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same set.

For this reason the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being non-contradictory,* the possibility of defining its concepts by a finite number of words,* or the practical certainty that it will never lead to a misunderstanding in human relations.†

As has been stated above, the formalist wishes to leave to the psychologist the task of selecting the "truly-mathe-

* See however Poincaré in *Scientia*, No. XXIV, p. 6.
† See however Borel in *Revue du Mois*, No. 80, p. 221.

matical" language from among the many symbolic languages that may be consistently developed. Inasmuch as psychology has not yet begun on this task, formalism is compelled to mark off, at least temporarily, the domain that it wishes to consider as "true mathematics" and to lay down for that purpose a definite system of axioms and laws of reasoning, if it does not wish to see its work doomed to sterility. The various ways in which this attempt has actually been made all follow the same leading idea, viz., the presupposition of the existence of a world of mathematical objects, a world independent of the thinking individual, obeying the laws of classical logic and whose objects may possess with respect to each other the "relation of a set to its elements." With reference to this relation various axioms are postulated, suggested by the practice with natural finite sets; the principal of these are: "*a set is determined by its elements*"; "*for any two mathematical objects it is decided whether or not one of them is contained in the other one as an element*"; "*to every set belongs another set containing as its elements nothing but the subsets of the given set*"; the axiom of selection: "*a set which is split into subsets contains at least one subset which contains one and not more than one element of each of the first subsets;* the axiom of inclusion: "*if for any mathematical object it is decided whether a certain property is valid for it or not, then there exists a set containing nothing but those objects for which the property does hold*"; the axiom of composition: "*the elements of all sets that belong to a set of sets form a new set.*"

On the basis of such a set of axioms the formalist develops now in the first place the theory of "finite sets." A set is called finite if its elements can not be brought into one-to-one correspondence with the elements of one of its subsets; by means of relatively complicated reasoning the principle of complete induction is proved to be a fundamental property of these sets;* this principle states that a property will be true for all finite sets if, first, it is true for all sets containing a single element, and, second, its validity for an arbitrary finite set follows from its validity for this same set reduced by a single one of its elements. That the formalist must give an explicit proof of this principle, which is self-evident for the

* Compare, e. g., Zermelo, "Sur les ensembles finis et le principe de l'induction complète," *Acta Mathematica*, 32, pp. 185–193.

finite numbers of the intuitionist on account of their construction, shows at the same time that the former will never be able to justify his choice of axioms by replacing the unsatisfactory appeal to inexact practice or to intuition equally inexact for him by a proof of the non-contradictoriness of his theory. For in order to prove that a contradiction can never arise among the infinitude of conclusions that can be drawn from the axioms he is using, he would first have to show that if no contradiction had as yet arisen with the nth conclusion then none could arise with the $(n + 1)$th conclusion, and secondly he would have to apply the principle of complete induction intuitively. But it is this last step which the formalist may not take, even though he should have proved the principle of complete induction; for this would require mathematical certainty that the set of properties obtained after the nth conclusion had been reached, would satisfy for an arbitrary n his definition for finite sets,* and in order to secure this certainty he would have to have recourse not only to the unpermissible application of a symbolic criterion to a concrete example but also to another intuitive application of the principle of complete induction; this would lead him to a vicious circle reasoning.

In the domain of finite sets in which the formalistic axioms have an interpretation perfectly clear to the intuitionists, unreservedly agreed to by them, the two tendencies differ solely in their method, not in their results; this becomes quite different however in the domain of infinite or transfinite sets, where, mainly by the application of the axiom of inclusion, quoted above, the formalist introduces various concepts, entirely meaningless to the intuitionist, such as for instance "*the set whose elements are the points of space*," "*the set whose elements are the continuous functions of a variable*," "*the set whose elements are the discontinuous functions of a variable*," and so forth. In the course of these formalistic developments it turns out that the consistent application of the axiom of inclusion leads inevitably to contradictions. A clear illustration of this fact is furnished by the so-called paradox of Burali-Forti.† To exhibit it we have to lay down a few definitions.

* Compare Poincaré, *Revue de Métaphysique et de Morale*, 1905, p. 834.
† Compare *Rendiconti del Circolo Matematico di Palermo*, 1897.

A set is called ordered if there exists between any two of its elements a relation of "higher than" or "lower than," with this understanding that if the element a is higher than the element b, then the element b is lower than the element a, and if the element b is higher than a and c is higher than b, then c is higher than a.

A well-ordered set (in the formalistic sense) is an ordered set, such that every subset contains an element lower than all others.

Two well-ordered sets that may be brought into one-to-one correspondence under invariance of the relations of "higher than" and "lower than" are said to have the same ordinal number.

If two ordinal numbers A and B are not equal, then one of them is greater than the other one, let us say A is greater than B; this means that B may be brought into one-to-one correspondence with an initial segment of A under invariance of the relations of "higher than" and "lower than." We have introduced above, from the intuitionist viewpoint, the smallest infinite ordinal number ω, i. e., the ordinal number of the set of all finite ordinal numbers arranged in order of magnitude.* Well-ordered sets having the ordinal number ω are called elementary series.

It is proved without difficulty by the formalist that an arbitrary subset of a well-ordered set is also a well-ordered set, whose ordinal number is less than or equal to that of the original set; also, that if to a well-ordered set that does not contain all mathematical objects a new element be added that is defined to be higher than all elements of the original set, a new well-ordered set arises whose ordinal number is greater than that of the first set.

We construct now on the basis of the axiom of inclusion the *set s which contains as elements all the ordinal numbers arranged in order of magnitude;* then we can prove without difficulty, on the one hand that s is a well-ordered set whose ordinal number can not be exceeded by any other ordinal number in magnitude, and on the other hand that it is possible, since not all mathematical objects are ordinal numbers, to create an

* The more general ordinal numbers of the intuitionist are the numbers constructed by means of Cantor's two principles of generation (compare *Math. Annalen*, vol. 49, p. 226).

ordinal number greater than that of s by adding a new element to s,—a contradiction.*

Although the formalists must admit contradictory results as mathematical if they want to be consistent, there is something disagreeable for them in a paradox like that of Burali-Forti because at the same time the progress of their arguments is guided by the principium contradictionis, i. e., by the rejection of the simultaneous validity of two contradictory properties. For this reason the axiom of inclusion has been modified to read as follows: "*If for all elements of a set it is decided whether a certain property is valid for them or not, then the set contains a subset containing nothing but those elements for which the property does hold.*"†

In this form the axiom permits only the introduction of such sets as are subsets of sets previously introduced; if one wishes to operate with other sets, their existence must be explicitly postulated. Since however in order to accomplish anything at all the existence of a certain collection of sets will have to be postulated at the outset, the only valid argument that can be brought against the introduction of a new set is that it leads to contradictions; indeed the only modifications that the discovery of paradoxes has brought about in the practice of formalism has been the abolition of those sets that had given rise to these paradoxes. One continues to operate without hesitation with other sets introduced on the basis of the old axiom of inclusion; the result of this is that extended fields of research, which are without significance for the intuitionist are still of considerable interest to the formalist. An example of this is found in the theory of potencies, of which I shall sketch the principal features here, because it illustrates so clearly the impassable chasm which separates the two sides.

* It is without justice that the paradox of Burali-Forti is sometimes classed with that of Richard, which in a somewhat simplified form reads as follows: "Does there exist a *least integer, that can not be defined by a sentence of at most twenty words?* On the one hand *yes*, for the number of sentences of at most twenty words is of course finite; on the other hand *no*, for if it should exist, it would be defined by the sentence of fifteen words formed by the words italicized above."

The origin of this paradox does not lie in the axiom of inclusion but in the variable meaning of the word "*defined*" in the italicized sentence, which makes it possible to define by means of this sentence an infinite number of integers in succession.

† Compare Zermelo, *Math. Annalen*, vol. 65, p. 263.

Two sets are said to possess the same potency, or power, if their elements can be brought into one-to-one correspondence. The power of set A is said to be greater than that of B, and the power of B less than that of A, if it is possible to establish a one-to-one correspondence between B and a part of A, but impossible to establish such a correspondence between A and a part of B. The power of a set which has the same power as one of its subsets, is called infinite, other powers are called finite. Sets that have the same power as the ordinal number ω are called denumerably infinite and the power of such sets is called aleph-null: it proves to be the smallest infinite power. According to the statements previously made, this power aleph-null is the only infinite power of which the intuitionists recognize the existence.

Let us now consider the concept: "denumerably infinite ordinal number." From the fact that this concept has a clear and well-defined meaning for both formalist and intuitionist, the former infers the right to create the "set of all denumerably infinite ordinal numbers," the power of which he calls aleph-one, a right not recognized by the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of denumerably infinite ordinal numbers can be built up in various ways, and second, that for every such set it is possible to assign a denumerably infinite ordinal number, *not* belonging to this set, the formalist concludes: "aleph-one is greater than aleph-null," a proposition, that has no meaning for the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist that it is impossible to construct* a set of denumerably infinite ordinal numbers, which could be proved to have a power less than that of aleph-one, but greater than that of aleph-null, the formalist concludes: "aleph-one is the second smallest infinite ordinal number," a proposition that has no meaning for the intuitionist.

Let us consider the concept: "real number between 0 and 1." For the formalist this concept is equivalent to "elementary

* If "construct" were here replaced by "define" (in the formalistic sense), the proof would *not* be satisfactory to the intuitionist. For, in Cantor's argument in *Math. Annalen*, vol. 49, it is not allowed to replace the words "können wir bestimmen" (p. 214, line 17 from top) by the words "muss es geben."

series of digits after the decimal point,"* for the intuitionist it means "law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations." And when the formalist creates the "set of all real numbers between 0 and 1," these words are without meaning for the intuitionist, even whether one thinks of the real numbers of the formalist, determined by elementary series of freely selected digits, or of the real numbers of the intuitionist, determined by finite laws of construction. Because it is possible to prove to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of real numbers between 0 and 1 can be constructed in various ways, and second that for every such set it is possible to assign a real number between 0 and 1, not belonging to the set, the formalist concludes: "the power of the continuum, i. e., the power of the set of real numbers between 0 and 1, is greater than aleph-null," a proposition which is without meaning for the intuitionist; the formalist further raises the question, whether there exist sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null, in other words, "whether the power of the continuum is the second smallest infinite power," and this question, which is still waiting for an answer, he considers to be one of the most difficult and most fundamental of mathematical problems.

For the intuitionist, however, the question as stated is without meaning; and as soon as it has been so interpreted as to get a meaning, it can easily be answered.

If we restate the question in this form: "Is it impossible to construct† infinite sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null?," then the answer must be in the affirmative; for the intuitionist can only construct denumerable sets of mathematical objects and if, on the basis of the intuition of the

* Here as everywhere else in this paper, the assumption is tacitly made that there are an infinite number of digits different from 9.

† If "construct" were here replaced by "define" (in the formalistic sense), and if we suppose that the problem concerning the pairs of digits in the decimal fraction development of π, discussed on p. 95, *can not be solved*, then the question of the text must be answered negatively. For, let us denote by Z the set of those infinite binary fractions, whose nth digit is 1, if the nth pair of digits in the decimal fraction development of π consists of unequal digits; let us further denote by X the set of all finite binary fractions. Then the power of $Z + X$ is greater than aleph-null, but less than that of the continuum.

linear continuum, he admits elementary series of free selections as elements of construction, then each non-denumerable set constructed by means of it contains a subset of the power of the continuum.

If we restate the question in the form: "Is it possible to establish a one-to-one correspondence between the elements of a set of denumerably infinite ordinal numbers on the one hand, and a set of real numbers between 0 and 1 on the other hand, both sets being indefinitely extended by the construction of new elements, of such a character that the correspondence shall not be disturbed by any continuation of the construction of both sets?," then the answer must also be in the affirmative, for the extension of both sets can be divided into phases in such a way as to add a denumerably infinite number of elements during each phase.*

If however we put the question in the following form: "Is it possible to construct a law which will assign a denumerably infinite ordinal number to every elementary series of digits and which will give certainty a priori that two different elementary series will never have the same denumerably infinite ordinal number corresponding to them?," then the answer must be in the negative; for this law of correspondence must prescribe in some way a construction of certain denumerably infinite ordinal numbers at each of the successive places of the elementary series; hence there is for each place c_ν a well-defined largest denumerably infinite number α_ν, the construction of which is suggested by that particular place; there is then also a well-defined denumerably infinite ordinal number α_ω, greater than all α_ν's and that can not therefore be exceeded by any of the ordinal numbers involved by the law of correspondence; hence the power of that set of ordinal numbers can not exceed aleph-null.

As a means for obtaining ever greater powers, the formalists define with every power μ a "set of all the different ways in which a number of selections of power μ may be made," and they prove that the power of this set is greater than μ. In particular, when it has been proved to the satisfaction of both

* Calling *denumerably unfinished* all sets of which the elements can be individually realized, and in which for every denumerably infinite subset there exists an element not belonging to this subset, we can say in general, in accordance with the definitions of the text: "*All denumerably unfinished sets have the same power.*"

formalist and intuitionist that it is possible in various ways to construct laws according to which functions of a real variable different from each other are made to correspond to all elementary series of digits, but that it is impossible to construct a law according to which an elementary series of digits is made to correspond to every function of a real variable and in which there is certainty a priori that two different functions will never have the same elementary series corresponding to them, the formalist concludes: "the power c' of the set of all functions of a real variable is greater than the power c of the continuum," a proposition without meaning to the intuitionist; and in the same way in which he was led from c to c', he comes from c' to a still greater power c''.

A second method used by the formalists for obtaining ever greater powers is to define for every power μ, which can serve as a power of ordinal numbers, "the set of all ordinal numbers of power μ," and then to prove that the power of this set is greater than μ. In particular they denote by aleph-two the power of the set of all ordinal numbers of power aleph-one and they prove that aleph-two is greater than aleph-one and that it follows in magnitude immediately after aleph-one. If it should be possible to interpret this result in a way in which it would have meaning for the intuitionist, such interpretation would not be as simple in this case as it was in the preceding cases.

What has been treated so far must be considered to be the negative part of the theory of potencies; for the formalist there also exists a positive part however, founded on the theorem of Bernstein: "If the set A has the same power as a subset of B and B has the same power as a subset of A, then A and B have the same power" or, in an equivalent form: "If the set $A = A_1 + B_1 + C_1$, has the same power as the set A_1, then it also has the same power as the set $A_1 + B_1$.

This theorem is self-evident for denumerable sets. If it is to have any meaning at all for sets of higher power for the intuitionist, it will have to be interpretable as follows: "If it is possible, *first* to construct a law determining a one-to-one correspondence between the mathematical entities of type A and those of type A_1, and *second* to construct a law determining a one-to-one correspondence between the mathematical entities of type A and those of types A_1, B_1, and C_1, then it is possible

to determine from these two laws by means of a finite number of operations a third law, determining a one-to-one correspondence between the mathematical entities of type A and those of types A_1 and B_1."

In order to investigate the validity of this interpretation, we quote the proof:

"From the division of A into $A_1 + B_1 + C_1$, we secure by means of the correspondence γ_1 between A and A_1 a division of A_1 into $A_2 + B_2 + C_2$, as well as a one-to-one correspondence γ_2 between A_1 and A_2. From the division of A_1 into $A_2 + B_2 + C_2$, we secure by means of the correspondence between A_1 and A_2 a division of A_2 into $A_3 + B_3 + C_3$, as well as a one-to-one correspondence γ_3 between A_2 and A_3. Indefinite repetition of this procedure will divide the set A into an elementary series of subsets $C_1, C_2, C_3, \ldots$, an elementary series of subsets $B_1, B_2, B_3 \ldots$, and a remainder set D. The correspondence γ_C between A and $A_1 + B_1$ which is desired is secured by assigning to every element of C_ν the corresponding element of $C_{\nu+1}$ and by assigning every other element of A to itself."

In order to test this proof on a definite example, let us take for A the set of all real numbers between 0 and 1, represented by infinite decimal fractions, for A_1 the set of those decimal fractions in which the $(2n-1)$th digit is equal to the $2n$th digit; further a decimal fraction that does not belong to A_1 will be counted to belong to B_1 or to C_1, according as the above-mentioned equality of digits occurs an infinite or a finite number of times. By replacing successively each digit of an arbitrary element of A by a pair of digits equal to it, we secure at once a law determining a one-to-one correspondence γ_1 between A and A_1. For of the element of A_1 that corresponds to an arbitrary well-defined element of A, such as, e. g., $\pi - 3$, we can determine successively as many digits as we please; it must therefore be considered as being well-defined.

In order to determine the element corresponding to $\pi - 3$ according to the correspondence γ_C, it is now necessary to decide first whether it happens an infinite or a finite number of times in the decimal fraction development of $\pi - 3$ that a digit in an odd-numbered place is equal to the digit in the following even-numbered place; for this purpose we should either have to invent a process for constructing an elementary series of such pairs of equal digits, or to deduce a contradiction

from the assumption of the existence of such an elementary series. There is, however, no ground for believing that either of these problems can be solved.*

Hence it has become evident that also the theorem of Bernstein, and with it the positive part of the theory of potencies, does not allow an intuitionistic interpretation.

So far my exposition of the fundamental issue, which divides the mathematical world. There are eminent scholars on both sides and the chance of reaching an agreement within a finite period is practically excluded. To speak with Poincaré: "Les hommes ne s'entendent pas, parce qu'ils ne parlent pas la même langue et qu'il y a des langues qui ne s'apprennent pas."

SHORTER NOTICES.

Essai de Géométrie analytique modulaire a deux Dimensions. By GABRIEL ARNOUX. Paris, Gauthier-Villars, 1911. xi + 159 pp.

WITH respect to a given prime m, modular space of two dimensions contains just m^2 distinct points (x, y), where $x, y = 0, 1, \cdots, m-1$. One identifies (x_1, y_1) with (x, y) if $x_1 \equiv x$, $y_1 \equiv y \pmod{m}$. The distance of (x, y) from the origin is an integer if $x^2 + y^2$ is a quadratic residue of m, but is a Galois imaginary if $x^2 + y^2$ is a non-residue. If we join the origin to one of our points and take the sine or cosine of the angle made with the x-axis, we obtain either an integer or a Galois imaginary modulo m; but the tangent is always an integer modulo m.

The set of points (x, y), where x, y range over all sets of integral solutions of $F(x, y) \equiv 0 \pmod{m}$, is called the modular curve $F \equiv 0$. The book under review is devoted chiefly to functions F of the first or second degree, the methods being analogous on the whole to those of ordinary analytic geometry. Homogeneous coordinates are not used.

* Such belief could be based only on an appeal to the principium tertii exclusi, i. e., to the axiom of the existence of the "set of all mathematical properties," an axiom of far wider range even than the axioms of inclusion, quoted above. Compare in this connection Brouwer, "De onbetrouwbaarheid der logische principes," *Tijdschrift voor Wijsbegeerte*, 2e jaargang, pp. 152–158.

There is one disadvantage of this theory as compared to the ordinary analytic geometry. In the latter case the curve uniquely determines the (algebraic) function. But obviously we can find as many polynomials in x, y as we please, each passing through the same given points, finite in number. The real points on a modular curve $F \equiv 0$ do not therefore form an adequate picture of $F \equiv 0$. To this end and for the purpose of investigating intersections and all but the most trivial questions, we must introduce also the imaginary points of $F \equiv 0$, i. e., solutions of $F(x, y) \equiv 0 \pmod{m}$ in which x (and likewise y) is a root of any congruence modulo m with integral coefficients. The aggregate of the resulting infinitude of points gives an adequate representation of the function. If the author had recognized this point of view and had succeeded in materializing a suitable graphical representation of this infinitude of points, he would have made a substantial contribution to modular geometry. But in confining himself to real points, the author goes no further than earlier writers.* The author and his collaborators G. Tarry and Laisant are apparently not familiar with the history of Galois imaginaries, as there is no mention of Galois when such imaginaries (of the second order) are used and since a particular case of Galois' generalization of Fermat's theorem is attributed on page 148 to Tarry.

L. E. DICKSON.

La Logique déductive dans sa dernière Phase de Développement. Par ALESSANDRO PADOA. Paris, Gauthier-Villars, 1912. 106 pp.

THIS treatise is an adaptation of a course of lectures given by the author at Geneva, under the auspices of the university. The author had previously lectured on the subject in Brussels, Pavia, Rome, Padua, Cagliari, and presented memoirs before the congresses at Rome, Leghorn, Parma, Padua, and Bologna. The treatise contains an explanation, with abundant examples, of the symbols of logic as used in the Formulario Matematico, of Peano, some study of their properties, analysis of their relations, and their reduction to a minimum number. The author expresses his point of view very well in the following:

* Veblen and Bussey, "Finite projective geometries," *Trans. Amer. Math. Soc.*, vol. 7 (1906), p. 241. As the title shows, these authors were interested only in definite finite geometries and not in general modular geometry, so that the criticism of Arnoux's text does not apply to them.

"I do not hope to suggest to you the sympathetic and touching optimism of Leibniz, who, prophesying the triumphal success of these researches, affirmed: 'I dare say that this is the last effort of the human mind, and, when this project shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect not less than the telescope serves to perfect their vision.' Although for some fifteen years I have given myself up to these studies, I have not a hope so hyperbolic; but I delight in recalling the candor of this master who, absorbed in scientific and philosophic investigations, forgot that the majority of men sought and continue to seek happiness in the feverish conquest of pleasure, money, and honors."

"Meanwhile we should avoid an excessive scepticism, because always and everywhere, there has been an élite—today less restricted than in the past—which was charmed by, and delights now in, all that raises one above the confused troubles of the passions, into the imperturbable immensity of knowledge, whose horizons become the more vast as the wings of thought become more powerful and rapid."

The symbols introduced are given below, with some examples. In the latter, N means integer, Np prime integer, $=$ (read *is the same thing as*) identity in the field of discussion, thus Rome = capital of Italy.

ϵ, *is*, or *is a*, appurtenance of individual to class. $7\epsilon Np$

$\subset$, *contains*

$\wedge$, *impossibility, absurdity*

$\vee$, *true*

$\supset$, *is contained in*, inclusion of subclass by a class. $15N \supset 3N$

$\smile$, *smallest superclass* $4N \smile 6N \supset 2N$

$\frown$, *largest common class* $4N \frown 6N \supset 12N$

ι, *class of a single element* $\iota 2 = Np \frown 2N$

$\imath$, *the only case* $2 = \imath(Np \frown 2N)$

$\ni$, *such that* $x \ni (x^2 + 26 = 10x) = \wedge$

$'$, *and*

$.\supset.$, *implies* $x\epsilon a . \supset . x\epsilon b : = : a \supset b$

two or more points may be used, thus $:\supset:$

$.\frown.$, *simultaneous affirmation*
$$2 < x < 7 . \frown . 4 < x < 9 : = : 4 < x < 7$$
$.\smile.$, *alternative affirmation*
$$2 < x < 7 . \smile . 4 < x < 9 : = : 2 < x < 9$$
$\sim$, *negation* of what follows to a stoppoint
$$\sim (8 + 3 = 10), \quad 6 \sim = Np$$
$\exists$, *there are some* $\exists[N^2 \frown (N^2 + N^2)]$

Cls, class of (taken in extension).

Elm, class of only one element.

After explaining the significance of these signs, the author considers the properties of certain logical relations, such as equality, appurtenance, inclusion, implication, etc. The possible transformations of logical statements are considered, the figures of the syllogism developed, and finally he shows that all the other symbols can be defined by means of three, namely $=$, $\frown$, and $\supset$, in other words, in terms of the notions of *identity*, *largest common subclass*, and *such that*. Although such definition is possible, it is inconvenient to use nothing but these symbols, so that the others ought to be retained for convenience. The style of the treatise is clear and very simple, and as an introduction to the study of mathematical logic, can scarcely be excelled.

JAMES BYRNIE SHAW.

Elements of Plane and Spherical Trigonometry. By JOHN GALE HUN and CHARLES RANALD MACINNES. New York, The Macmillan Company, 1911. vii + 101 pp., with tables, pp. 102–205.

IN writing this book, the authors have undertaken "to present in as brief and clear a manner as possible the essentials of a short course in trigonometry." This aim they appear to have kept constantly before them. The language of the book is simple, concise, and interesting. The subject matter is brief enough to be covered by a class in somewhat less time than that usually required, and still comprehensive enough to take in all that is usually regarded as essential.

The authors have included one subject not often treated in a text book in trigonometry, namely, the drawing of graphs of equations in polar coordinates. For this, they give the

following two reasons: "Firstly, because such problems aid in giving the student a clearer idea of the way in which the trigonometric functions vary as the angle is changed, and secondly, because of a very common lack of sufficient knowledge of polar coordinates on the part of students beginning the study of calculus." This work is introduced early in the book—earlier, indeed, than seems advisable, as it precedes the treatment of the relation of the functions of angles differing by 90° and 180°, a working knowledge of which would render the process of graphing far less difficult.

In the explanation of logarithms and the use of tables of logarithms and trigonometric functions, the authors have avoided the rather common error of being so brief as to be fully intelligible only to one already understanding the subject. Their treatment is clear and explicit, supplemented by problems for the student to solve.

CORA B. HENNEL.

NOTES.

THE twentieth annual meeting of the American Mathematical Society will be held in New York City on Tuesday and Wednesday, December 30–31, 1913. At this meeting Professor H. B. FINE will deliver his Presidential Address, on "An unpublished theorem of Kronecker respecting numerical equations." Titles and abstracts of papers intended for presentation at the annual meeting should be in the hands of the Secretary by December 13.

THE opening (September) number of volume 15 of the *Annals of Mathematics* contains the following papers: "Singular point transformations in two complex variables," by G. R. CLEMENTS; "On the projective differential geometry of plane anharmonic curves," by S. W. REAVES; "On the rank of a symmetrical matrix," by L. E. DICKSON; "Note on the rank of a symmetrical matrix," by J. H. M. WEDDERBURN; "On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$," by R. D. CARMICHAEL.

THE concluding (October) number of volume 35 of the *American Journal of Mathematics* contains the following

papers: "On the four-dimensional angles of the semiregular polytopes of S_4," by P. H. SCHOUTE; "A generalization of Volterra's derivative of a function of a curve," by C. A. FISCHER; "On differential invariants," by J. B. SHAW; "Some properties of closed convex curves in a plane," by A. EMCH; "Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors," by L. E. DICKSON; "Even abundant numbers," by L. E. DICKSON; "The identical relations between the elements of any oblique triple system of surfaces," by H. D. THOMPSON; "Transformations and invariants connected with linear homogeneous difference equations and other functional equations," by CORA B. HENNEL; "Projective differential geometry of rational cubic curves," by J. A. NYBERG.

To the list of American doctorates conferred in 1913 which appeared in the October BULLETIN, should be added, T. H. BROWN, Yale, "The effect of radiation on a small particle revolving about Jupiter"; and H. N. WRIGHT, California, "On a tabulation of reduced binary quadratic forms of a negative determinant."

THE following courses in mathematics are announced in the several technical schools during the first semester 1913–1914:

BERLIN.—By Professor E. LAMPE: Differential and integral calculus, with exercises, eight hours; Definite integrals and differential equations, two hours.—By Professor G. HETTNER: Differential and integral calculus, with exercises, eight hours; Differential equations, two hours.—By Professor G. SCHEFFERS: Descriptive geometry, ten hours.—By Professor O. KRIGAR-MENZEL: Mechanics, four hours; Acoustics, two hours; Theory of equilibrium, four hours.—By Professor G. WALLENBERG: Selected chapters of elementary mathematics, four hours; Theory of functions, two hours; Theory of potential, two hours.—By Professor ST. JOLLES: Descriptive geometry, ten hours; Graphical statics, with exercises, four hours.—By Professor K. HAENTZSCHEL: Elements of the calculus, with exercises, six hours; Mechanics, four hours.

BRUNSWICK.—By Professor R. DEDEKIND: Elements of the theory of numbers, two hours; Introduction to the calculus

of probability, one hour.—By Professor R. FRICKE: Analytic geometry and algebra, four hours; Differential and integral calculus, with exercises, six hours.—By Professor H. E. TIMERDING: Algebra, two hours; Geometry of position, two hours; Descriptive geometry with exercises, ten hours.—By Professor A. WERNICKE: Statics of rigid bodies, with exercises, six hours.—By Professor W. SCHLINK: Technical mechanics, II, with exercises, five hours.

CLAUSTHAL.—By Professor H. MOHRMANN: Higher mathematics and mechanics, six hours; Descriptive geometry, four hours.

DELFT.—By Professor J. A. BARRAU: Determinants and introduction to the calculus, three hours; Theory of projection, with exercises, four hours; Applications of descriptive geometry, three hours.—By Professor W. H. L. JANSSEN VAN RAAY.—Advanced algebra and the calculus, five hours; Advanced calculus, four hours.—By Professor W. A. VERSLUYS: Introduction to analytic geometry, two hours; Analytic geometry of space, two hours.—By Professor J. CARDINAAL: Methods of projection, four hours; Surfaces and space curves, four hours.—By Professor G. SCHOUTEN: Mechanics, four hours; Kinematics, four hours.

DRESDEN.—By Professor G. HEGER: Plane cubic curves, one hour.—By Professor G. HELM: Higher mathematics, IV, with exercises, four hours; Theory of potential, three hours; Colloquium, two hours.—By Professor M. KRAUSE: Higher mathematics, II, with exercises, six hours; Advanced algebra, four hours; Seminar, two hours.—By Professor W. LUDWIG: Descriptive geometry, with exercises, seven hours; Perspective, one hour; Analytic geometry of quadric surfaces, three hours; History of mathematics in antiquity, one hour.

KARLSRUHE.—By Professor A. KRAZER: Mathematics, I, with exercises, eight hours.—By Professor R. FUETER: Higher mathematics, II, three hours; Partial differential equations with applications, two hours.—By Professor M. DISTELI: Descriptive geometry, with exercises, eight hours; Graphical statics, with exercises, two hours.—By Dr. W. VOGT: Analytic geometry of the plane and of space, three hours; Projective geometry, two hours.—By Dr. F. NOETHER: Elements of mechanics, with exercises, four hours; Theory of elasticity,

two hours.—By Dr. O. HAUPT: Arithmetic and algebra, with exercises, three hours; Plane and spherical trigonometry, with exercises, three hours; Exercises in the principles of higher mathematics, two hours.

STUTTGART.—By Professor HALLER: Plane and spherical trigonometry, with exercises, four hours.—By Professor E. STÜBLER: Lower analysis, four hours; Differential and integral calculus, four hours.—By Professor W. KUTTA: Higher mathematics, II, with exercises, eight hours; Seminar, two hours.—By Professor E. WÖLFFING: Theory of functions, I, three hours.—By Professor R. MEHMKE: Descriptive geometry, with exercises, seven hours; Graphical calculation, with exercises, three hours; Seminar, two hours.—By Professor K. KOMMERELL: Foundations of geometry, two hours.

ZURICH.—By Professor A. HIRSCH: Higher mathematics, I, five hours; Higher mathematics, III, four hours.—By Professor FRANEL: Higher mathematics, I, five hours; Higher mathematics, III, four hours.—By Professor H. WEYL: Analytic geometry, with exercises, six hours.—By Professor M. GROSSMANN: Descriptive geometry, with exercises, eight hours; Projective geometry, four hours; Seminar, two hours.—By Professor L. KOLLROS: Descriptive geometry, with exercises, eight hours; Projective geometry, four hours; Seminar, two hours.—By Professor A. HURWITZ: Algebraic equations, four hours; Seminar (with Professor Weyl), two hours.—By Dr. L. KIENAST: Theory of functions of a complex variable, two hours.—By Dr. A. KRAFT: Theory of extension and vector analysis, four hours.

MASS. INSTITUTE OF TECHNOLOGY.—By Professor F. S. WOODS: Advanced calculus and differential equations, four hours.—By Professor F. H. BAILEY: Fourier's series, two hours.—By Professor E. B. WILSON: Analytic mechanics, two hours; Relativity, two hours.

DR. W. BLASCHKE, of the University of Greifswald, has been appointed associated professor of mathematics in the German technical school at Prague.

DR. F. RUSCH, of the University of Zurich, has been appointed professor of mathematics and physics at the University of Tientsin.

DR. H. TIETZE, of the German technical school at Brünn, has been promoted to a full professorship of mathematics.

PROFESSOR G. FABER, of the University of Königsberg, has been appointed professor of mathematics in the University of Strassburg.

DR. L. BIEBERBACH, of the University of Königsberg, has been appointed professor of mathematics in the University of Basel.

PROFESSOR GINO LORIA, of the University of Genoa, has been elected to membership in the Halle academy of sciences.

AT the annual meeting of the royal academy dei Lincei, of Rome, Professor G. BAGNERA, of the University of Palermo, was elected to corresponding membership; Professor I. FREDHOLM, of the University of Stockholm, Dr. G. W. HILL, of Nyack, N. Y., and Professor A. HURWITZ, of the technical school at Zürich, were elected foreign members.

PROFESSOR F. KLEIN, of the University of Göttingen, has been elected foreign member of the academy of sciences of Naples and corresponding member of the Berlin academy of sciences.

DR. H. CHATELET has been appointed professor of mechanics at Toulouse, to succeed Professor J. DRACH, who has accepted a similar position at the University of Paris.

DR. R. GARNIER has been appointed professor of mathematics at the University of Poitiers, as successor to Professor P. BOUTROUX, now of Princeton University.

PROFESSOR A. L. BOWLEY, of University College, Reading, has resigned.

PROFESSOR W. H. YOUNG, of Liverpool University, has been appointed Hardinge professor of mathematics at the University of Calcutta, for the purpose of organizing the school of higher mathematics. As his new duties require his residence in India only part of the time, he will retain his present professorship in Liverpool University.

AT Princeton University Professor H. B. FINE has returned from a year's leave of absence and resumed his duties as head of the department of mathematics and dean of the departments of science. Professor OSWALD VEBLEN is abroad on leave of absence for the year.

AT Ohio Wesleyan University, Professor G. N. Armstrong, who has been abroad on leave of absence, has resumed his academic duties.

PROFESSOR L. A. HOWLAND, of Wesleyan University, has been promoted to a full professorship of mathematics.

AT Rutgers College, Professor RICHARD MORRIS has been promoted to the professorship of mathematics, and appointed head of the department. Dr. W. B. STONE, of the University of Michigan, has been appointed assistant professor of mathematics.

PROFESSOR F. CAJORI, of Colorado College, has received the honorary degree of doctor of laws from the University of Colorado and the honorary degree of doctor of science from the University of Wisconsin.

PROFESSOR S. E. RASOR, of Ohio State University, has been promoted to a full professorship of mathematics.

DR. W. C. KRATHWOHL has been appointed associate professor of mathematics in Ripon College.

AT Lehigh University Dr. J. B. REYNOLDS has been promoted to an assistant professorship of mathematics; Mr. M. S. KNEBELMAN has been appointed instructor in mathematics.

PROFESSOR L. C. PLANT, of the University of Montana, has accepted the professorship of mathematics at the Michigan Agricultural College. Dr. N. J. LENNES, of Columbia University, succeeds Professor Plant at the University of Montana.

DR. S. LEFSCHETZ, of the University of Nebraska, has been appointed instructor in mathematics in the University of Kansas.

MR. G. W. MULLINS and Mr. J. A. NORTHCOTT have been appointed instructors in mathematics in Columbia University.

MR. F. W. DARLING has been appointed instructor in mathematics at Cornell University.

MR. H. R. KINGSTON has been appointed instructor in mathematics at the University of Manitoba.

MISS M. MCDONALD has accepted the professorship of mathematics at Oxford College for Women, Oxford, Ohio.

DR. JOSEPHINE E. BURNS has been appointed instructor in mathematics at the University of Illinois.

PROFESSOR C. BOURLET, of the Conservatoire des Arts et Métiers in Paris, died August 7, at the age of 47 years.

M. GASTON TARRY died at Havre, June 21, 1913, at the age of 70 years.

THE death is announced of Dr. G. ROTH, emeritus professor of mathematics in the University of Strassburg.

PROFESSOR ALEXANDER MACFARLANE, formerly of the University of Edinburgh, the University of Texas, and Lehigh University, died August 28, at the age of 62 years. He had been a member of the American Mathematical Society since 1891.

PROFESSOR J. R. EASTMAN, professor of mathematics in the U. S. Navy from 1865 to 1898, died September 26, at the age of seventy-seven years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ATTI del III congresso della Mathesis, società italiana di matematica, Genova, ottobre, 1912. Roma, Manuzio, 1913. 8vo. 127 pp.

BACKES (W.). Ein Beweis des Fermatschen Satzes. Mainz, Lehrlingshaus, 1913. 8vo. 4 pp. M. 2.00

BLISS (G. A.) and KASNER (E.) The Princeton colloquium lectures on mathematics: Fundamental existence theorems; Differential-geometric aspects of dynamics. 8vo. New York, American Mathematical Society, 1913. 7 + 107 + 117 pp. $1 50

BÜTZBERGER (F.). Ueber bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion. Leipzig, Teubner, 1913. 8vo. 60 pp. M. 1.50

CORNUBERT (R.). Dictionnaire allemand-français et français-allemand des termes et locutions scientifiques. Chimie, physique, mathématiques, mineralogie. Paris, Dunod et Pinat, 1913. 8vo. 252 pp. Cartonné. Fr. 9.00

CRUEWELL (E. R.). Die Regeln des Dreiecks. 5te vermehrte Ausgabe. Diese Ausgabe enthält den Beweis für den Satz von Fermat. Berlin, Buschhardt, 1913. 8vo. 4+119 pp. M. 10.00

DARBOUX (G.). See LEBON (E.).

DAVIS (E. W.). The calculus. Edition de luxe. New York, Macmillan, 1913. 12mo. 21+383+63 pp. Flexible leather. $2.40

EULER (L.). Opera omnia. Series I: Opera mathematica. Vol. XX: Commentationes analyticae ad theoriam integralium ellipticorum pertinentes. Edidit A. Krazer. Volumen prius. Leipzig, Teubner, 1912. 8vo. 12+372 pp. M. 28.00

HAHN (H.). See SCHOENFLIES (A.).

HILBERT (D.). Grundlagen der Geometrie. 4te, durch Zusätze und Literaturhinweise von neuem vermehrte und mit 7 Anhängen versehene Auflage. (Wissenschaft und Hypothese, Band VII.) Leipzig, Teubner, 1913. 8vo. 6+258 pp. Cloth. M. 6.00

JAKOB (H.). Zum grossen Fermatschen Satz. Beweis. Dresden, Köhler, 1913. 8vo. 4 pp. M. 0.50

JOURDAIN (P. E. B.). The nature of mathematics. (People's books.) New York, Dodge, 1913. 16mo. 4+92 pp. $0.20

KASNER (E.). See BLISS (G. A.).

KRAZER (A.). See EULER (L.).

LAISANT (C. A.). Mathematics. (Thresholds of science.) London, Constable, 1913. 8vo. 166 pp. 2s.

LEBON (E.). Gaston Darboux. Biographie; bibliographie analytique des écrits. 2e édition, entièrement refondue. (Savants du jour.) Paris, Gauthier-Villars, 1913. 8vo. 4+96 pp. Fr. 7.00

MANSION (P.). Abriss der Theorie der Hyperbelfunktionen nebst einer rein analytischen Theorie der Kreisfunktionen. Nach der 3ten

französischen Auflage übersetzt. Leipzig, Teubner, 1913. 8vo. 44 pp. M. 1.25

——. Précis de la théorie des fonctions hyperboliques, suivi d'une théorie purement analytique des fonctions circulaires. 3e édition revue et augmentée. Paris, Gauthier-Villars, 1913. 8vo. 44 pp. Fr. 1.25

MIKAMI (Y.). See SMITH (D. E.).

OLIVIER (L. E.). Solution du problème de Fermat. Paris, Gauthier-Villars, 1913. 8vo. 4 pp. Fr. 0.75

O'SHAUGHNESSY (L.). Integrability of the differential equation representing the sum of a family of series. Philadelphia, O'Shaughnessy, 1912. 4to. 23 pp.

PORTIER (B.). Sur les panmagiques de module 8, à grilles. Exposition pratique. Paris, Gauthier-Villars, 1913. 8vo. 4 pp. Fr. 0.50

PUSTAU (W.). Lösung des grossen Fermatschen Satzes. 2te Bearbeitung. Frankfurt, Knauer, 1913. 8vo. 12 pp. M. 2.00

QUANDT (J.). Kulturelle Bedeutung der Mathematik. (Festrede.) Pirna, Diller, 1913. 8vo. 16 pp. M. 0.30

RADLIK (K.). Der Beweis des grossen Fermatschen Satzes und der Nachweis fundamentaler Irrtümer in der mathematischen Lehre. Geschrieben als Herausforderung aller Universitäten von einem krassen Aussenseiter der Wissenschaft. Berlin, Selbstverlag, 1913. 8vo. 2+27 pp. M. 4.00

RIPPAS (W. DE). En confirmation du principe erroné en mathématique. Paris, Dunod et Pinat, 1913. 8vo. Fr. 3.00

ROUSSOTTE (H.). Recueil d'exercices sur le calcul différentiel. Paris, Dunod et Pinat, 1913. 8vo. Fr. 2.50

SALOMON (C.). Nouveaux essais de magie arithmétique polygonale. Paris, Gauthier-Villars, 1913. 8vo. 28 pp. Fr. 1.00

SCHOENFLIES (A.) und HAHN (H.). Die Entwicklung der Lehre von Punktmannigfaltigkeiten. Bericht. 1ter Teil. (*Jahresbericht der Deutschen Mathematiker-Vereinigung. 8ter Band, 2tes Heft.*) 2te Auflage. Leipzig, Teubner, 1913. 8vo. 7+390 pp. M. 8.00

SCHOEDER (H.). Die Zentralflächen der Paraboloide und Mittelpunktsflächen 2ten Grades. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von Martin Schillung. Neue Folge. Nr. 10.) Leipzig, Schilling, 1913. 8vo. 72 pp. M. 2.00

SMITH (D. E.) and MIKAMI (Y.). A history of Japanese mathematics. Chicago, Open Court Co., 1913. Cloth. $3.00

SUINI (A.). Sul vizio d'origine delle geometrie non euclidee. Piacenza, Porta, 1913. 8vo. 8 pp. L. 1.00

WEITZEL (C. G.). Unterrichts-Briefe zur Einführung in die höhere Mathematik. 2te–8te Lieferung. Wien, Hartleben, 1913. M. 3.50

WEYL (H.). Die Idee der Riemannschen Fläche. (Mathematische Vorlesungen an der Universität Göttingen. Nr. V.) Leipzig, Teubner, 1913. 8vo. 10+170 pp. Cloth. M. 8.00

II. ELEMENTARY MATHEMATICS.

ABHANDLUNGEN über den mathematischen Unterricht in Deutschland. 1ter Band: Die höhere Schulen in Norddeutschland. 12+102, 7+204, 5+118, 6+93, and 12+183 pp. M. 18.00. 2ter Band: Die höhere Schulen in Süddeutschland und Mitteldeutschland. 5+85, 12+78, 4+104, 4+48, 6+51, 4+18, 6+58, and 6+49 pp. M. 14.00. Leipzig, Teubner, 1913. 8vo.

BARLOW. Tables des carrés, cubes, racines carrées, racines cubiques et inverses. Paris, Dunod et Pinat, 1913. 8vo. Fr. 5.00

BAUER (W.) und HANXLEDEN (E. v.). Lehrbuch der Mathematik. Oberstufe der Geometrie. Braunschweig, Vieweg, 1913. 8vo. 10+286 pp. Cloth. M. 4.50

BAYLISS (R. W.). A first formal geometry. London, Arnold, 1913. 8vo. 160 pp. 1s. 6d.

BECK (H.). Lehrstoff für den Raumlehre-Unterricht. Halle, Schrödel, 1913. 8vo. 72 pp. M. 0.75

BLUMER (S.). Raumlehre. Aarau, Trüb, 1913. 8vo. 76 pp. M. 1.75

BÖTTCHER (R.) und SENDLER (R.). Raumlehre. 1ter Teil: Planimetrie. 4te, verbesserte Auflage. Breslau, Handel, 1912. 8vo. 4+99 pp. M. 1.30

CATANIA (S.). Algebra elementare. Parte I. 3a edizione, rifatta. Catania, Giannotta, 1913. 16mo. 8+170 pp. L. 2.00

CHANTICLAIRE. Ce qu'il faut savoir pour calculer rapidement de tête. Paris, Dunod et Pinat, 1913. 16mo. Cartonné. Fr. 0.60

COLLINS (J. V.). Advanced algebra. New York, American Book Co., 1913. 12mo. 10+342 pp. Cloth. $1.00

DAVIDSON (J.). Arithmetic and algebra. (Self-educator series.) New York, Doran, 1913. 12mo. 11+235 pp. $0.75

DEAKIN (R.). A new school geometry. Part 2. London, Mills & Boon, 1913. 8vo. 138 pp. 1s. 6d.

ECKHARDT (O.). Stoff und Methode des mathematischen Unterrichts in Knabenmittelschulen. Münster, Aschendorff, 1913. 8vo. 7+170 pp. Cloth. M. 2.80

EMMERICH (A.). Leitfaden und Uebungsbuch der Stereometrie. Weinheim, Ackermann, 1913. 8vo. 8+160 pp. M. 2.20

FENKNER (H.). Arithmetische Aufgaben. Ausgabe A. Teil IIa. 5te Auflage. Berlin, Salle, 1913. 8vo. 4+234 pp. M. 3.00

FITE (W. B.). First course in algebra. Boston, Heath, 1913. 12mo. 7+285 pp. Cloth. $0.90

GEIPEL (G.) und WEGEMANN (G.). Lehrbuch der Mathematik für Oberlyzeen. 2ter Teil: Klasse II. Bielefeld, Velhagen & Klasing, 1913. 8vo. 5+160 pp. M. 1.80

GIESSEN (S.). Raumlehre. Ausgabe D. Münster, Coppenrath, 1913. 8vo. 61 pp. M. 0.60

GOLLER (I.). Raumlehre mit geometrischem Zeichnen. Ausgabe A. 3te Auflage. Stuttgart, Bonz, 1913. 8vo. 48 pp. M. 0.40

HANXLEDEN (E. v.). See BAUER (W.).

HOVESTADT (H.). See KILLING (W.).

INSKIP (G. D.). Tables of squares, logarithms, etc. 4th edition, enlarged. Philadelphia, Inskip, 1913. 12mo. 5+5+447+161 pp. Cloth. $1.50

KEMPTHORNE (W. B.). See WILLIAMS (W. H.).

KENT (F. C.). A first course in algebra. New York, Longmans, 1913. 12mo. 7+249+13 pp. Cloth. $1.00

KILLING (W.) und HOVESTADT (H.). Handbuch des mathematischen Unterrichts. 2ter Band. Leipzig, Teubner, 1913. 8vo. 10+472 pp. Cloth. M. 11.00

KNOBLAUCH (C.). Multiplikator "Rechenfix." Multiplikationen 1–1000×1–100. 2te Auflage. Dresden, Der Baumarkt, 1913. 3+100 pp. Cloth. M. 6.50

LESSER (O.). See SCHWAB (K.).

LÖRCHER (O.) und LÖFFLER (E.). Geometrie und Trigonometrie. Stuttgart, Grub, 1913. 8vo. 12+300 pp. Cloth. M. 3.60

MANCHESTER (R. E.). The teaching of mathematics. Syracuse, N. Y., Bardeen, 1913. 12mo. 75 pp. $0.50

MARTUS (H. C. E.). Mathematische Aufgaben zum Gebrauche in den obersten Klassen höheren Lehranstalten. 4ter Teil. 3te Auflage. Dresden, Koch, 1913. 8vo. 245 pp. Cloth. M. 5.60

MATRICULATION mathematics papers, University of London, from June, 1907, to June, 1913. London, Clive, 1913. 8vo. 112 pp. Sewed. 1s. 6d.

NEUHAUS (O.). Geheimnisse des Schnellrechnens, u. s. w. 6te Auflage. Papiermühle, Vogt, 1913. 8vo. 3+56+7 pp. M. 1.50

NOODT (G.). Mathematische Unterrichtsbücher. Ausgabe B. 2tes Heft. 2te Auflage. Bielefeld, Velhagen & Klasing, 1913. 8vo. 3+96 pp. M. 1.10

OEHLER (H.). See RICHTER (M.).

PAGLIANO (C.). Sunti di algebra. Rocca S. Casciano, Cappelli, 1913. 8vo. 85 pp. L. 0.50

PALATINI (F.). Aritmetica ed algebra. 2a edizione. Torino, Gallizio, 1913. 8vo. 330 pp. L. 4.00

PYRKOSCH. Lehrbuch der Mathematik für die Oberstufe höherer Lehranstalten. Bielefeld, Velhagen & Klasing, 1913. 8vo. 12+512 pp. Cloth. M. 4.20

REINHARD (P.). Rechnungstabelle. Tableau de calcul. 4te Auflage. Bern, Francke, 1913. M. 5.00

REPETITORIEN des mathematischen und sprachlichen Unterrichtsstoffes der Gymnasien. 5ter Band. 2ter Teil: Algebra. 3+122 pp. M. 1.50. 3ter Teil: Algebra. 3+100 pp. M. 125. Langensalza, Schulbuchhandlung, 1913. 8vo.

RICHTER (M.) und OEHLER (H.). Algebra. 1ter Teil: Klasse V. Stuttgart, Bonz, 1913. 8vo. 140 pp. Cloth. M. 2.00

RÜEFLI (J.). Ebene Geometrie. 7te Auflage. Bern, Francke, 1913. 8vo. 4+106 pp. M. 1.15

SCHUBERT (H.). Vierstellige Tafeln für logarithmisches Rechnen, in zwei Farben zusammengestellt. Neue Ausgabe von R. Haussner. (Samm-

lung Göschen, Neue Auflage, Nr. 81.) Berlin, Göschen, 1913. 8vo. 175 pp. Cloth. M. 0.90

SCHWAB (K.) und LESSER (O.). Geometrie. Ausgabe A. 1ter Teil. 3te Auflage. Leipzig, Freytag, 1913. 8vo. 290 pp. Cloth. M. 3.60

SENDLER (R.). See BÖTTCHER (R.).

SEYFFARTH (W.). Elementarmathematik. 4ter Teil: Stereometrie. Dresden-Blasewitz, Bleyl & Kaemmerer, 1913. 8vo. 4+112 pp. M. 2.00

SMITH (F. K.). Computer; formerly "Eby's handy tables." Revised, enlarged, improved, specialized. Pleasanton, Kan., Smith, 1913. 32mo. 305 pp. Cloth. $2.50

SMOLEY (C.). Five-decimal logarithmic-trigonometric tables, etc. 2d edition, revised and enlarged. New York, McGraw-Hill, 1913. 12mo. 3+174 pp. $0.50

SUPPANTSCHITSCH (R.). Geometrie, Trigonometrie und analytische Geometrie. Wien, Tempsky, 1913. 8vo. 300 pp. Cloth. M. 4.00

VERONESE (G.). Nozioni elementare di geometria intuitiva. 4a edizione. Padova, Drucker, 1912. 8vo. 108 pp. L. 1.50

VIELLEFOND (A.). Précis de géométrie. Paris, Dunod et Pinat, 1913. 16mo. Cartonné. Fr. 3.50

VILLA (A.). Geometria analitica del piano e sue applicazioni. Milano, Sonzogno, 1913. 16mo. 63 pp. L. 0.20

——. Trigonometria sferica e sue applicazioni. Milano, Sonzogno, 1913. 16mo. 60 pp. L. 0.20

WEGEMANN (G.). See GEIPEL (G.).

WILLIAMS (W. H.) and KEMPTHORNE (W. B.). First course in algebra. Chicago, Lyons and Carnahan, 1913. 12mo. 4+346 pp. Cloth. $1.00

III. APPLIED MATHEMATICS.

ALLIAUME (M.). Eléments d'astronomie. Paris, Gauthier-Villars, 1913. 8vo. 12+333 pp. Fr. 5.00

ANDREWS (E. S.). Further problems in the theory and design of structures. An advanced textbook. London, Chapman & Hall, 1913. 8vo. 244 pp. 7s. 6d.

BAUDOIN (P.). See BOURLET (C.).

BLONDLOT (R.). Einführung in die Thermodynamik. Deutsche Ausgabe besorgt von C. Schorr und F. Platschek. Dresden, Steinkopff, 1913. 8vo. 8+102 pp. M. 4.00

BOUASSE (H.). Cours de thermodynamique. 2e édition, complètement transformée et considérablement augmentée. 1re partie: Principes généraux; gaz et vapeurs. Tome 2e du Cours de Physique. Paris, Delagrave, 1913. 8vo. 462 pp. Broché. Fr. 18.00

BOURLET (C.) et BAUDOIN (P.). Leçons de géometrie et éléments de géométrie descriptive. Paris, Dunod et Pinat, 1913. 16mo. Cartonné. Fr. 4.00

CAMPBELL (N. R.). Moderne Elektrizitätslehre. Uebersetzt von U. Meyer. Dresden, Steinkopff, 1913. 8vo. 11+423 pp. Cloth. M. 15.50

CAMPBELL (W. W.). Stellar motions; with special reference to motions determined by means of the spectrograph. New Haven, Conn., Yale University, 1913. 8vo. 10+328 pp. Cloth. $4.00

CARMICHAEL (R. D.). The theory of relativity. (Mathematical monograph series.) New York, Wiley, 1913. 8vo. 74 pp. Cloth. $1.00

CHAMBERS (G. F.). Astronomy. New York, Van Nostrand, 1913. 16mo. 335 pp. $1.50

CLAUZEL (G.). Méthode géométrique permettant d'établir simplement plusieurs équations importantes de mécanique. Paris, Dunod et Pinat, 1913. 4to. 32 pp. Fr. 2.00

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